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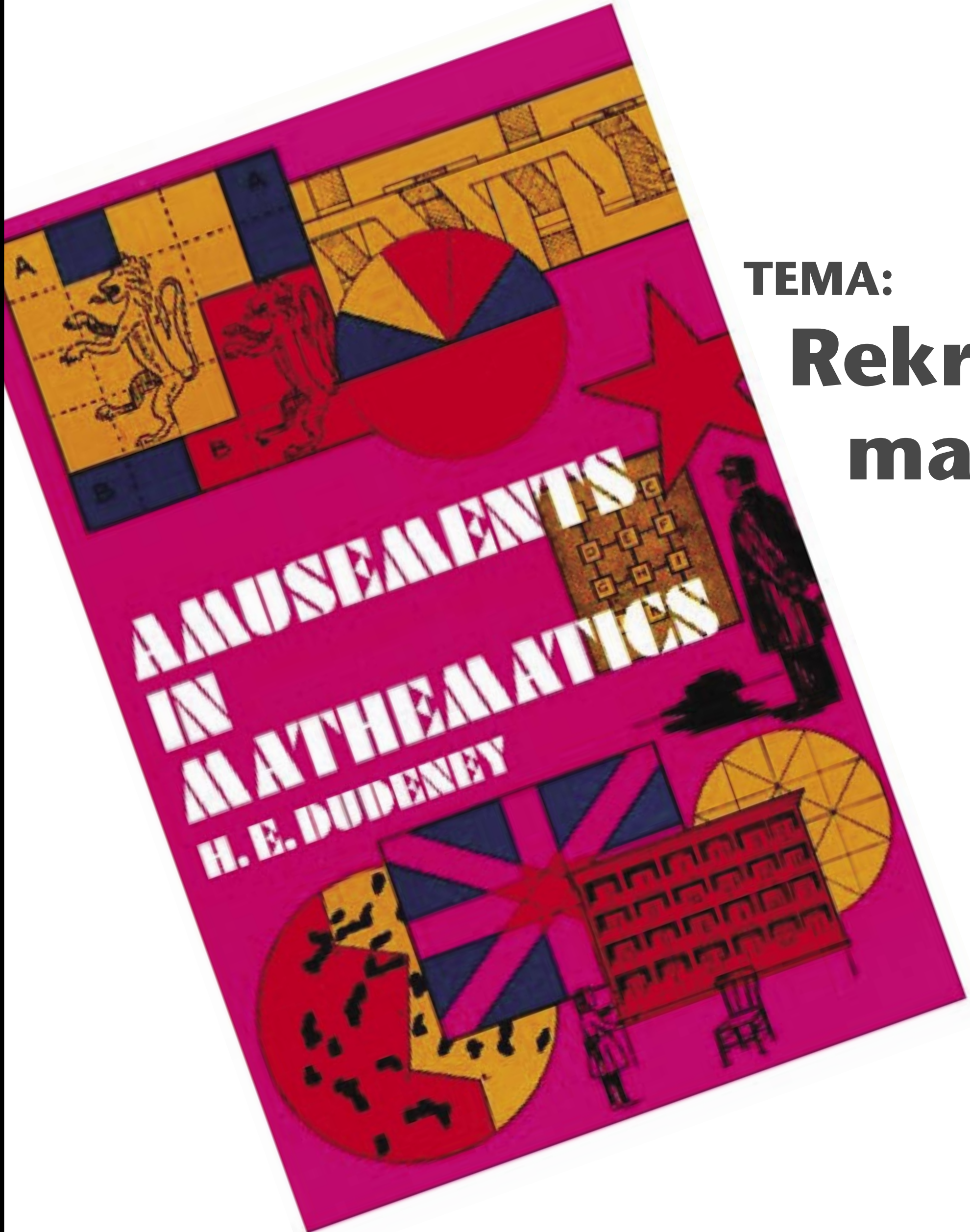
MATILDE



TEMA:
Rekreativ matematik

NYHEDSBREV FOR DANSK MATEMATISK FORENING

NR 36 DECEMBER 2008

Leder

Af: Bent Ørsted

Dette nummer af Matilde har som tema, rekreativ matematik, hvorved vi tænker på den umiddelbare og underholdende side af matematikken. Faktisk har vi jo allerede på bladet en trofast og fantasifuld redaktør i Mogens Esrom Larsen, der bidrager med sin kolumne Aftermath - hvor man finder mange spændende problemer. Disse er ofte både sjove og overraskende, og belyser den væsentlige side af matematik der handler om at løse gåder, ofte med snedige vinkler og tricks (men som prof. Svend Bundgaard sagde: når man først en gang har set et trick, så er det ikke længere et trick men en metode).

Vi har hentet nogle artikler til dette nummer fra tidsskriftet, Mathematical Intelligencer - og er taknemmelige til Springer for velvilje hertil. Især er det en fornøjelse at bringe en klassiker af Martin Gardner, der jo er berømt for sine artikler om matematik i Scientific American igennem mange år. Se her omtalen af Martin Gardner i Mogens Esrom Larsens artikel længere fremme.

Det kan naturligvis være svært at angive det skarpe skel mellem rekreativ matematik og "almindelig" matematik; man ville gerne og helst udbrede det spændende og underholdende ved faget til så mange som muligt, at bibringe glæden ved at opleve og indse matematiske sammenhænge og kendsgerninger - men desværre kræver det jo af og til en del tålmodighed. Her kommer små og store matematiske gåder til undsætning; og det tiltaler mange, se blot på udbredelsen af Sudoku eller andre indslag som f. eks. diverse intelligenstests, hvor man skal finde (Mensa-problemer) den næste figur i rækken, for ikke at tale om spil som f. eks. Hex (af Piet Hein, se bare Google). Her er i øvrigt en lille sekvensgåde: Hvad er det næste i rækken

11, 12, 13, 14, 15, 16, 17, 20, 22, 24, 31?

Vink: Hvad kommer før?

Som det fremgår af nogle af de følgende artikler, så er (igen) begrebet sandsynlighed centralt, og er da også helt med i daglige oplevelser af matematik. Spil og chancer, odds og tendenser er naturlige dele af den moderne hverdag. Måske lidt for naturlige, i hvert fald hvad angår den financielle gamblen med store værdier. Men lad det nu ligge, sammen med leasing-karuseller og anden anvendt aritmetik.

Som sagt kan gåder være en sjov og uforpligtende indgang til matematik, tag nu f. eks. historien om to russiske søstre, der boede i hver sin by. På samme tid en morgen startede de hver mod den anden by; klokken 12 middag mødtes de og fortsatte, hver i sit konstante tempo. Klokken 16 var den ene søster fremme (ved den anden by) og klokken 21 var den anden fremme. Hvad tid startede de den morgen?

Matematikken hjælper os til at tænke over mange forskellige problemer, både af opdigtet og reel natur, og kan gøre det usynlige synligt; vi må være taknemmelige for både de små og de store glimt af indsigt.

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Indhold:

Forsidebilledet er H.E. Dudeneys klassiske bog "Amusements in Mathematics", der også udkom på dansk i 1949 under titlen "Tænke tænke tænke." Bogen er frit tilgængelig på nettet i Projekt Gutenberg http://www.gutenberg.org/etext/16713

Rekreativ matematik - en selvmodsigelse?

Af: Mogens Esrom Larsen
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"A serious business" – en alvorlig affære? – var titlen på David Singmasters tiltrædelsesforelæsning til sit professorat ved University of The South Bank i 1992. Så vidt jeg ved det eneste professorat i netop emnet "rekreativ matematik," der ellers har været en hobby for matematikere eller en forretning for amatører. Det sidste var i Davids tanker.

Fremtrædende matematikere har dyrket emnet, Édouard Lucas, W. W. Rouse Ball, H. S. M. Coxeter, Roger Penrose, John H. Conway, David Gale, Ian Stewart oma, mens amatørerne tæller de gamle, H. A. Dudeney i England, Sam Loyd og Matin Gardner i USA; sidstnævnte høstede uvisnelig hæder for sin månedlige side i Scientific American 1955–80. Derfor har der været holdt møder – "Gatherings for Gardner" hvert andet år siden 1994 i Atlanta.

Hvorfor nu det? Det skyldes, at skolen i USA behandler især matematik meget stedmoderligt; der lægges stor vægt på fagets kedeligste side: Løsning af problemer, – der ikke kan ophidse nogen, – ved brug af foreskrevne skabeloner, – som eleverne ikke fatter relevansen af. Der findes derfor en række især amerikanske matematikere, som skylder Martins månedlige indsprøjtning i Scientific American, at de overhovedet kom til at interessere sig for faget. Og de mødes så i Atlanta for at hylde deres guru!

Men for Dudeney og Sam Loyd var det "business." De to var højt elskede bidragydere til aviser og blade af opgaver, der kunne udfordre menigmand. Og hvad er så en ideel opgave? Den skal jo kunne forstås af enhver, og i princippet løses af enhver, – ikke med matematik, men med intelligens! At den kan løses matematisk er kun til opgavestillerens fordel, hvis han da ikke selv er superintelligent.

I dette nr. af Matilde stiller jeg en række typiske opgaver, men også to af opgaverne i sidste nr. er typiske. Opgaven om tryllekunsten er helt perfekt! Selv garvede matematikere klarer sig ikke uden en god idé! Og opgaven om duellanterne er velegnet, mens sokkerne, der fører til Pells ligning, nok er for svær. Men den simpleste løsning kan alle finde. De nye opgaver er alle "rekreative," efter mit skøn. Da jeg stillede jeep–opgaven i Illustreret Videnskab, fik jeg et begejstret takkebrev fra Norge; det var gået op for læseren, at der ikke er grænser for, hvor langt flåden kan nå ud i ørkenen. (Den harmoniske rækkes divergens.) Se, det er jo et eksempel på noget helt andet: Kan jeg lokke læseren til at begejstres over sin nyerhvervede indsigt?

Opgaven med de tre kraftværker og de tre huse skyldes i øvrigt Dudeney, – mærkeligt nok, Euler burde have haft den. (Den kommer måske i et endnu ikke trykt bind af hans værker?)

En lille personlig anekdote. David Singmaster spurgte mig i 1990, om der var diophantiske trekanter, (dvs. med heltallige sider), der opfyldte vinkelrelationen, at $\angle A = 2(\angle B - \angle C)$? Den var ikke i litteraturen! Det "oplagte" svar var trekanten med siderne 6, 7 og 8. En rekreativ opgave? Måske.

Jeg har haft fornøjelsen at fortælle denne historie for AMS–MAA i San Francisco, for Matematiklærerforeningen i Odense, som kollokvium i Ålborg og sågar i studenterkollokviet i København!

Nogle andre rekreative problemer er dem, der handler om pentominoer, dvs. figurer dannet af 5 lige store kvadrater, så figurerne er sammenhængende side ved side eller for skakbrikken tårnet. Der er 12 sådanne, der ikke er indbyrdes kongruente. De blev lanceret af Dudeney for 100 år siden. Spørgsmålet om antallet af måder, de kan dække et skakbræt uden centrum blev besvaret på MIT i 1959 ved hjælp af en computer fra IBM, svaret er 65. Dette blev omtalt af Martin Gardner i 1960. Solomon Golumb, University of Southern California, stillede opgaven i 1965 at vise, at en "ruder" dvs, et kvadrat på spidsen, med 61 mindre kvadrater med akseparallelle sider, ikke kan dækkes af pentominoerne, således at det tomme kvadrat er i centrum. Jeg viste i 1986, at de 12 figurer kun kan anbringes i ruden sådan, at det tomme kvadrat er på randen. Af dem kender jeg 10 muligheder, men jeg ved ikke, om det er samtlige. Rekreativt? Det synes jeg!

Men hvad er så rekreativ matematik? Tja, enhver spontan glæde ved en problemløsning. Og det er ikke så forskelligt fra al anden matematik. Vi har jo den erfaring, at et problem skal abstraheres ud af sin sammenhæng, så der ses bort fra distraherende irrelevante sider, løses så generelt som muligt, og så vendes tilbage mod sin oprindelse til sidst. Og der kan gå lang tid. Tænk på tidsforskellen fra Euklids og Apollonios' keglesnit i 2. årh. fvt. og Johannes Keplers model for marsbanen i 1609!

Et slående eksempel af ældre dato er Yale 4662, en assyrisk lertavle fra ca. 1800 fvt. Her stilles og besvares opgaven at bestemme siderne i et rektangel, beskrevet som en gravet grøft, hvor man kender produktet, altså grøftens areal, og "summen af længde og bredde!" Men det er jo ikke nogen naturlig problemstilling, det er ren glæde over at kunne løse problemet. At denne løsning har haft mange senere anvendelser er vel uimodsigeligt.

Også de assyriske tabeller over pythagoræiske tripler er slående. Fx. Plimpton 322, der godt nok starter med (3,4,5), men ender med (13500, 12709, 18541). De har begejstret regnet videre på formlerne, så langt energien slog til!

For nogle år siden blev verden udsat for et spørgsmål. Et offer får at vide, at der bag tre døre er en bil og to gedebukke hhv. Kan han gætte døren med bilen, må han få den i præmie, – ellers bliver han stanget! Nu peger han på en dør, men i stedet for at åbne den, åbner studieværten en anden dør og fremviser en ged. Offeret får nu tilbudt at vælge om, altså vælge den sidste dør eller at fastholde sit oprindelige valg. Utallige er de matematikere, der sagde, at det kunne være lige meget. Erfaringen er, at opgaver med betingede sandsynligheder går hen over hovedet på de fleste.

I 1976 fandt Ernö Rubik i Budapest på et pædagogisk hjælpemiddel, der skulle træne hans arkitektstuderendes rumfornemmelse. En terning delt i 27 ens terninger, som med en genial konstruktion kunne drejes om terningens centrum i de tre hovedretninger. Rubiks terning gik sin sejrsgang over verden, gjorde Singmaster verdenskendt og indtog H C Ørsted Instituttets kantine til jul 1980 på

Det er Rubiks cube på billedet

Flemming Topsøes foranledning. For at udnytte terningens matematiske potentiale skrev jeg et lille notat på 8 sider om dens gruppeteori, der forklarer løsningsmetoderne. Som den eneste dansksprogede vejledning blev den i kopier spredt overalt. Det endte med, at jeg udgav en lille bog om terningen i juni 1981, – den solgte 25000 eksemplarer på 3 mdr.

I 1985 havde Chr. U. Jensen med hjem fra USA en variant med $64 = 4^3$ terninger, kaldet "Rubik's Revenge," som han forærede mig. Der var ingen opskrift, så jeg bestemte dens gruppe og publicerede løsningen i Am. Math. Monthly i juli 1986. Selv om det ikke er min mest citerede artikel, kandiderer den til at være min mest læste. (Den konkurrerer med artiklen om pentominoerne på ruderen. Der udkom en piratoversættelse på kinesisk!)

Det viste sig, at i vor forstand er Rubik's Revenge simplere end den oprindelige terning, idet gruppen er et direkte produkt af to undergrupper, der hver for sig er simplere end gruppen for Rubiks terning! (Som ikke kan spaltes i et direkte produkt.) Det betyder, at det er lige meget, hvilke terninger man først ordner, hvad der jo ikke er oplagt.

En pudsighed bør nævnes. 100 år før red 15–klodsspillet verden som en mare. Det er en 2–dimensional permutationsopgave, hvor man lægger 15 kvadrater på et større kvadrat, der er 4×4 kvadrater stort. Nu må man skyde et kvadrat over på den ubesatte plads. Man kan kun udføre lige permutationer, hvis den tomme plads skal begynde og ende samme sted, typisk i nederste højre hjørne. Hvis man derfor lægger de nummererede kvadrater tilfældigt op, er der sandsynligheden $\frac{1}{2}$ for, at de kan ordnes i rækkefølge, 1–15. Det, at nogle blev desperate over umuligheden, fik Julius Petersen til at forsøge at forklare permutationers fortegn i Illustreret Tidende. Og det bar frugt. Da jeg som barn blev præsenteret for legetøjet hos min nabo, fortalte han, at hvis man byttede om på to af brikkerne, så kunne puslespillet ikke længere løses.

Lad mig slutte med at udfordre læseren! Binomialkoefficienter er som alle hele tal enten lige eller ulige. Et spørgsmål, hvis nytteværdi fortaber sig i tågerne, men hvis løsning kræver subtil matematik, er: Hvad er sandsynligheden for, at en tilfældig binomialkoelfficent er (u)lige?



Lucky Numbers and 2187

Martin Gardner

When I was told that Martin Gardner had submitted an article for the Entertainments column, I tried to remember when I had heard his name for the first time. I failed: it seemed that Gardner's books had always been there. I remember my high school friends playing the game of Hex or following the evolution of Life; we learned about these games (as well as many other things) from Russian translations of Gardner's books.

I have on my bookshelf two editions of the Russian translation of his book "Mathematics: Magic and Mystery"; the second edition (1967) was printed in 100,000 copies; the fifth one (1986) in 700,000 copies. I don't know whether Gardner got a cent of royalties from Soviet publishers, but the deep gratitude of millions of his Soviet readers is unquestionable.

Let me thank Dr. Gardner for his existence—and for sending an article for this column!

This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

Dette er en direkte gengivelse af en artikel fra Mathematical Intelligencer, vol 19, nummer 2, 1997.

Please send all submissions to the Mathematical Entertainments Editor, **Alexander Shen,** Institute for Problems of Information Transmission, Ermolovoi 19, K-51 Moscow GSP-4, 101447 Russia; e-mail:shen@landau.ac.ru

The house where I grew up as a child in Tulsa, Oklahoma, has an address of 2187 S. Owasso. Of course I never forgot this number. Many years ago, when I was visiting my imaginary friend Dr. Irving Joshua Matrix, the world's most famous numerologist, I asked him if there was anything remarkable about 2187.

He immediately replied: "It is 3 raised to the power of 7. If you write it in base 3 notation it is 10,000,000."

"I'm amazed you would know that!" I exclaimed. "Anything else unusual about 2187?"

"My dear chap," Dr. Matrix responded with a heavy sigh, "*every* number has endless unusual properties. Exchange the last two digits of 2187 to make 2178, multiply by 4, and you get 8712, the second number backward. Take 2187 from 9999 and the result is 7812, its reversal. Multiplying 21 by 87 produces 1827, the same digits in a different order. And have you noticed that the first four digits of the constant *e*, 2718, and the number of cubic inches in a cubic foot, $12^3 = 1728$, are each permutations of 2187? You might ask your readers how quickly they can insert plus or minus signs inside 2187 to make the expression add to zero."

I was struggling to jot all this down on my notepad when Dr. Matrix added: "And 2187 is, of course, one of the lucky numbers."

I had never heard of lucky numbers. What follows is a summary of what I learned about them from Dr. Matrix, and from the references listed at the end of this article.

The notion of lucky numbers originated about 1955 with Stanislaw Ulam, the great Polish mathematician who co-invented the H-bomb and was the father of cellular automata theory. It is one of the most studied types of what are called "sieve numbers." The oldest, most important sieve numbers are the primes. They are called sieve numbers because they can be generated by what is known as the Sieve of Erathosthenes.

> Imagine all the positive integers written in counting order. Cross out all multiples of 2, except 2. The next uncrossed-out number is 3. Cross out all multiples of 3, except 3. Continue in this way, sieving out multiples of 5, 7, 11, and so on. The numbers that remain (except for the special case of 1) are the primes.

The sieving process is slow and tedious, but if continued to infinity it will identify every prime.

Using a sieve for generating lucky numbers is similar. Curiously, it produces numbers closely related to primes even though they are mixtures of primes and composites (nonprimes). Here is how the procedure works.

> Step 1: Cross out every second number: 2, 4, 6, 8, . . . , leaving only the odd integers.

Step 2: Note that the second uncrossed-out integer is 3. Cross out every third number not yet eliminated: 5, 11, 17, 23, ...
Step 3: The third surviving number from the left is 7. Cross out every seventh integer not yet crossed out: 19, 39, ...
Step 4: The fourth number from the left is 9. Cross out every ninth number not yet eliminated, starting with 27.

As you continue in this fashion you will see that certain integers permanently escape getting killed. Ulam called them "lucky numbers." Figure 1 lists all the luckies less than 1,000.

Eratosthenes's sieve abolished all numbers except the primes. The procedure is based on division. Ulam's sieve, on the contrary, is based entirely on a number's *position* in the counting series. Using Eratosthenes's sieve you have to count every integer as you go along. Using Ulam's sieve you count only the integers not previously eliminated.

Although the luckies are identified by a sieving process completely different from Eratosthenes's sieve, the amazing thing is that luckies share many properties with primes. The density of luckies in a given interval among the counting numbers is extremely close to the density of primes in the same interval. For example, there are 25 primes less than 100, and 23 luckies less than 100. The overall asymptotic density for each type of number is the same!

The distances between successive primes and the distances between successive luckies keep growing longer as the numbers grow in size. These distances also are almost the same for both number types. The number of twin primes—primes that differ by 2—is close to the number of twin luckies. There are eight twin primes less than 100, and seven twin luckies in the same interval. Although primes play a much more significant role in number theory than luckies, the similarities suggest that many of the properties of primes are less unique than previously assumed. Their properties may be more a product of sieving than anything else!

The most notorious unsolved problem involving primes, now that Fermat's last Theorem has been proved, is the Goldbach conjecture. It states that every even number greater than 2 is the sum of two primes. There is a similar unsolved conjecture about luckies: that every even number is the sum of two luckies. This has been computer tested for integers up to 100,000, and perhaps further than that, in recent years, without finding an exception.

In a 1996 booklet about number problems, Charles Ashbacher, of Cedar Rapids, Iowa, conjectures that every lucky number appears at the tail of a larger lucky. For example, 7 is at the end of 37; 9 at the end of 49; 15 at the end of 615; and so so on. Lucky 87 is at the end of my old house number 2187. Lucky 579 is at the end of lucky 96579. Ashbacher wrote a computer program that verified his conjecture for 22 of the first 100 luckies. This suggests, he writes, that his conjecture is a good bet.

It is easy to determine if certain large numbers are not lucky. Consider 98765. We can quickly tell it is not lucky because it has a digital root of 8. The digital root of a number is its equivalence modulo 9—that is, the remainder when divided by 9. If there is no remainder, the digital root is 9. A digital root is quickly obtained by adding the digits of a number, then adding again if the sum has more than one digit, and continuing this way until just one digit remains. Lucky numbers display all digital roots except 2, 5, and 8. Why? Because all numbers with those three digital roots have the form $3k + 2$ and the first two sieving steps eliminate them all.

Dr. Matrix called my attention to the curious fact that 13, considered the most unlucky of all numbers, is the fifth lucky, the sixth prime, and the seventh Fibonacci number.

A few weeks after my meeting with Dr. Matrix I received from him a fax message listing the following identities:

2187 + 1234 = 3421
2187 + 12345 = 14532
2187 + 123456 = 125643
2187 + 1234567 = 1236754
2187 + 12345678 = 12347865
2187 + 123456789 = 123458976

Note how the sums on the right are permutations of the numbers added to 2187.

It has been proved that no polynomial formula will generate only primes, and I would guess that the same is true for the luckies. However, simple quadratic formulas will generate sequences of primes and luckies. One way to search for such formulas was invented by Ulam. On a square grid write the integers in a spiral fashion as shown in Figure 2, and indicate the luckies by color. Note that nine luckies clump along a diagonal. Applying the calculus of finite differences to these luckies we discover that they are generated by $4x^2 + 2x + 1$, as x takes the values $-3, -2, -1, 0, 1, 2, 3, 4, 5$.

The spiral can start with any higher number to reveal clumps along different diagonals. Leonhard Euler found that $x^2 + x + 41$ generates forty primes by letting x take values 0 through 39. If you write the integers in a spiral starting with 41, these primes will fill the entire diagonal of a 40 × 40 grid! Is there a quadratic formula equally rich, or perhaps even richer, in finding a clump of luckies? I will be interested

```
1 3 7 9 13 15 21 25 31 33 37 43 49 51 63
67 69 73 75 79 87 93 99 105 111 115 127 129 133 135
141 151 159 163 169 171 189 193 195 201 205 211 219 223 231
235 237 241 259 261 267 273 283 285 289 297 303 307 319 321
327 331 339 349 357 361 367 385 391 393 399 409 415 421 427
429 433 451 463 475 477 483 487 489 495 511 517 519 529 535
537 541 553 559 577 579 583 591 601 613 615 619 621 631 639
643 645 651 655 673 679 685 693 699 717 723 727 729 735 739
741 745 769 777 781 787 801 805 819 823 831 841 855 867 873
883 885 895 897 903 925 927 931 933 937 957 961 975 979 981
991 993 997
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Figure 1. A computer printout of lucky numbers less than 1,000, supplied by Charles Ashbacher. Note that '99 will be a lucky year.

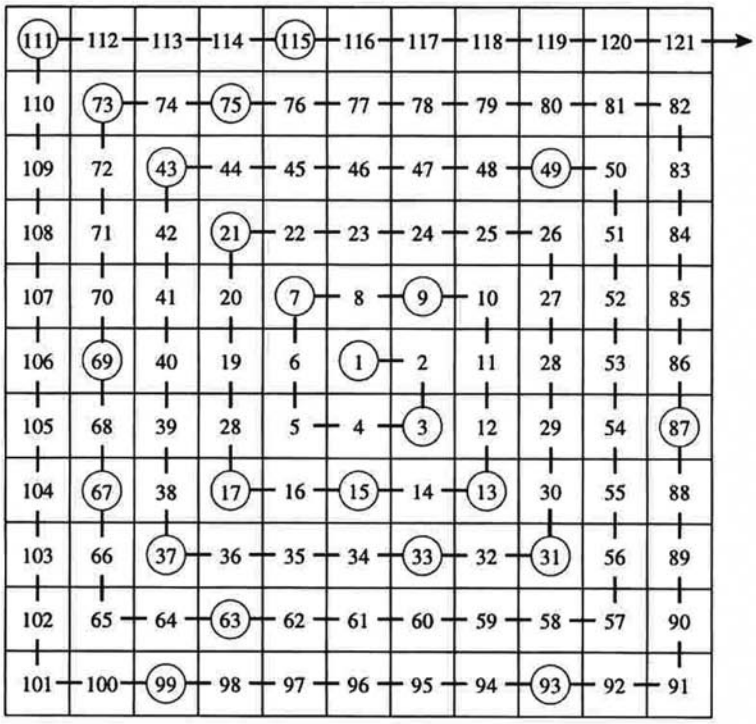

Figure 2. Ulam's spiral technique for finding Quadratic lucky-rich formulas.

in hearing from any reader who finds such a formula.

There is a classic proof by Euclid that there is an infinity of primes. Although it is easy to show there is also an infinity of lucky numbers, the question of whether an infinite number of luckies are primes remains, as far as I know, unproved. Also unsolved is whether there is an infinity of twin luckies.

Dr. Matrix enjoys practical jokes. When we talked about 2187 he pointed out that if this number is divided by 9999 the quotient is .218721872187. . . . I was momentarily surprised until I realized that *any* integer of n digits, not made entirely of nines, when divided by a number consisting of n nines, produces a decimal fraction in which the original number is repeated endlessly as the quotient's period.

"Ulam discovered lucky numbers with his lucky imagination," Dr. Matrix added. "Note the letters at positions 2, 1, 8, and 7 in LUCKY IMAGINATION. What do they spell?"

The first three lucky numbers are 1, 3, and 7. Now 137 not only is a prime but it is one of the most interesting of all three-digit numbers. It is, of course, the notorious fine-structure constant, the most mysterious of all constants in physics. I mentioned this to Dr. Matrix. This prompted him to talk for twenty minutes about 137. Here are some highlights of what he said:

Check the King James Bible's first chapter, third verse, and seventh word. The word is "light." Dr. Matrix reminded me that the fine-structure constant is intimately connected with light.

The reciprocal of 137, or 1 divided by 137, produces the decimal fraction 007299270072992700. . . . The period is a palindrome!

Partition 137 into 13 and 7. The thirteenth letter of the alphabet is M and the seventh is G—my two initials!

Chlorophyll, which takes light from the sun to give energy to plants, is made of exactly 137 atoms.

Dr. Matrix asked me to write my old house number twice, 21872187, and put the number into my hand calculator. This number, he informed me, is exactly divisible by 137. I performed the division and sure enough, the readout displayed the integer 159651. I got an even greater surprise when Dr. matrix asked me to turn the calculator upside down. The number was the same inverted!

Dr. Matrix next asked me to divide 159651 by 73. The result was 2187! I later discovered that this was another of Dr. Matrix's hoaxes. *Any* number of the form of ABCDABCD is evenly divisible by 137 and 73. The reason? ABCDABCD is the product of ABCD and 10001. The two prime factors of 10001 are 137 and 73, so dividing ABCABCD by those two numbers will naturally restore ABCD. Of course the quotient after the first division is not likely to be invertible.

"Is there any connection," I asked, "between the lucky numbers and 666, the famous number of the Beast in New Testament prophecy?"

Dr. Matrix put his fingertips together and closed his emerald eyes for a full minute before he spoke.

"Consider your old house number 2187, and the first four luckies 1, 3, 7, 9. Omit the 1 in each number to leave 287 and 379. Add the two numbers and you get 666. By the way, I forgot to mention earlier that if you divide 18, the middle digits of 2187, by 27, the first and last digits, the quotient is .66666666. . . . ."

I conclude with a mind-reading trick of my own that involves 2187. Ask someone to put this number into a calculator's display. With your back turned, tell him to multiply it by any number he likes without revealing this number to you. He next calls out, in any order, each digit in the product except one nonzero digit. You at once name the missing digit.

How do you do it? As he calls out digits, keep adding them in your head until you know the digital root of their sum. This is easily done by casting out nines as explained earlier. If the digital root is 9, he omitted 9. If less than 9, he left out a digit equal to the difference between 9 and the digital root. For example, if the digital root is 2, he omitted 7.

I leave it to you to figure out why this always works with 2187. Hint: 2187 has a digital root of 9.

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"On Certain Sequences of Integers Defined by Sieves." Verna Gardiner, R. Lazarus, Nicholas Metropolis, and Stanislaw Ulam. *Mathematics Magazine*, Vol. 31, 1956, pp. 117–122.

forsættes side 10

Michael Kleber:

The best card trick

forsat fra side 9

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You, my friend, are about to witness the best card trick there is. Here, take this ordinary deck of cards, and draw a hand of five cards from it. Choose them deliberately or randomly, whichever you prefer - but do not show them to me! Show them instead to my lovely assistant, who will now give me four of them, one at a time: the 7♠, then the Q♡, the 8♣, the 3♢. There is one card left in your hand, known only to you and my assistant. And the hidden card, my friend, is the K♠.

Surely this is impossible. My lovely assistant passed me four cards, which means there are 48 cards left that could be the hidden one. I did receive a little information: the four cards came to me one at a time, and by varying that order my assistant could signal one of 4! = 24 messages. It seems the bandwidth is off by a factor of two. Maybe we are passing one extra bit of information illicitly? No, I assure you: the only information I have is a sequence of four of the cards you chose, and I can name the fifth one.

The Story

If you haven't seen this trick before, the effect really is remarkable; reading it in print does not do it justice. (I am forever indebted to a graduate student in one audience who blurted out "No way!" just before I named the hidden card.) Please take a moment to ponder how the trick could work, while I relate some history and delay giving away the answer for a page or two. Fully appreciating the trick will involve a little information theory and applications of the Birkhoff-von Neumann theorem and Hall's Marriage theorem. One caveat, though: fully appreciating this article involves taking its title as a bit of showmanship, perhaps a personal opinion, but certainly not a pronouncement of fact!

The trick appeared in print in Wallace Lee's book "Math Miracles" [1] in which he credits its invention to William Fitch Cheney, Jr., a.k.a. "Fitch". Fitch was born in San Francisco in 1894, son of a professor of medicine at Cooper Medical College, which later became the Stanford Medical School. After receiving his B.A. and M.A. from the University of California in 1916 and 1917, Fitch spent eight years working for the First National Bank of San Francisco and then as statistician for the Bank of Italy. In 1927 he earned the first math Ph.D. ever awarded by MIT; it was supervised by C.L.E. Moore and entitled "Infinitesimal deformation of surfaces in Riemannian space." Fitch was an instructor and assistant professor in mathematics at Tufts from 1927 until 1930, and thereafter a full professor and sometimes department head, first at the University of Connecticut until 1955 and then at the University of Hartford (Hillyer College before 1957) until his retirement in 1971; he remained an adjunct until his death in 1974.

For a look at his extra-mathematical activities, I am indebted to his son Bill Cheney, who writes:

My father, William Fitch Cheney, Jr., stage-name "Fitch the Magician," first became interested in the art of magic when attending vaudeville shows with his parents in San Francisco in the early 1900s. He devoted countless hours to learning slight-of-hand skills and other "pocket magic" effects with which to entertain friends and family. From the time of his initial teaching assignments at Tufts College in the 1920s, he enjoyed introducing magic effects into the classroom, both to illustrate points and to assure his students' attentiveness. He also trained himself to be ambidextrous (although naturally left-handed), and amazed his classes with his ability to write equations simultaneously with both hands, meeting in the center at the "equals" sign.

Each month the magazine M-U-M, official publication of the Society of American Magicians, includes a section of new effects created by society members, and "Fitch Cheney" was a regular by-line. A number of his contributions have a mathematical feel. His series of seven "Mental Dice Effects" (beginning Dec. 1963) will appeal to anyone who thinks it important to remember whether the numbers 1, 2, 3 are oriented clockwise or counterclockwise about their common vertex on a standard die. "Card Scense" (Oct. 1961) encodes the rank of a card (possibly a joker) using the fourteen equivalence classes of permutations of $abcd$ which remain distinct if you declare $ac = ca$ and $bd = db$ as substrings: the card is placed on a piece of paper whose four edges are folded over (by the magician) to cover it, and examining the creases gives precisely that much information about the order in which they were folded [2].

[1] Published by Seeman Printery, Durham, N.C., 1950; Wallace Lee's Magic Studio, Durham, N.C., 1960; Mickey Hades International, Calgary, 1976.

[2] This sort of "Purloined Letter"-style hiding of information in plain sight is a cornerstone of magic. From that point of view, the "real" version of the five-card trick secretly communicates the missing bit of information; Persi Diaconis tells me there was a discussion of ways to do this in the late 1950s. For our purposes we'll ignore these clever but non-mathematical ruses.

While Fitch was a mathematician, the five card trick was passed down via Wallace Lee's book and the magic community. (I don't know whether it appeared earlier in M-U-M or not.) The trick seems to be making the rounds of the current math community and beyond thanks to mathematician and magician Art Benjamin, who ran across a copy of Lee's book at a magic show, then taught the trick at the Hampshire College Summer Studies in Mathematics program [3] in 1986. Since then it has turned up regularly in "brain teaser" puzzle-friendly forums; on the rec.puzzles newsgroup, I once heard that it was posed to a candidate at a job interview. It made a recent appearance in print in the "Problem Corner" section of the January 2001 Emissary, the newsletter of the Mathematical Sciences Research Institute, and as a result of writing this column I am learning about a slew of papers in preparation that discuss it as well. It is a card trick whose time has come.

The Workings

Now to business. Our "proof" of impossibility ignored the other choice my lovely assistant gets to make: which of the five cards remains hidden. We can put that choice to good use. With five cards in your hand, there are certainly two of the same suit; we adopt the strategy that the first card my assistant shows me is of the same suit as the card that stays hidden. Once I see the first card, there are only twelve choices for the hidden card. But a bit more cleverness is required: by permuting the three remaining cards my assistant can send me one of only 3! = 6 messages, and again we are one bit short.

The remaining choice my assistant makes is which card from the same-suit pair is displayed and which is hidden. Consider the ranks of these cards to be two of the numbers from 1 to 13, arranged in a circle. It is always possible to add a number between 1 and 6 to one card (modulo 13) and obtain the other; this amounts to going around the circle "the short way." In summary, my assistant can show me one card and transmit a number from 1 to 6; I increment the rank of the card by the number, and leave the suit unchanged, to identify the hidden card.

It remains only for me and my assistant to pick a convention for representing the numbers from 1 to 6. First totally order a deck of cards: say initially by rank, $A23...JQK$, and break ties by ordering the suits in bridge (= alphabetical) order, $\clubsuit\,\diamondsuit\,\heartsuit\,\spadesuit$. Then the three cards can be thought of as smallest, middle, and largest, and the six permutations can be ordered, e.g., lexicographically [4].

Now go out and amaze (and illuminate [5]) your friends. But please: just make sure that you and your own lovely assistant agree on conventions and can name the hidden card flawlessly, say 20 times in a row, before you try this in public. As we saw above, it's not hard to name the hidden card half the time - and it's tough to win back your audience if you happen to get the first one wrong. (I speak, sadly, from experience.)

The Big Time

Our scheme works beautifully with a standard deck, almost as if four suits of thirteen cards each were chosen just for this reason. While this satisfied Wallace Lee, we would like to know more. Can we do this with a larger deck of cards? And if we replace the hand size of five with n, what happens?

First we need a better analysis of the information passing. My assistant is sending me a message consisting of an ordered set of four cards; there are $52 \times 51 \times 50 \times 49$ such messages. Since I see four of your cards and name the fifth, the information I ultimately extract is an unordered set of five cards, of which there are $\binom{52}{5}$ which for comparison we should write as $52 \times 51 \times 50 \times 49 \times 48/5!$. So there is plenty of extra space: the set of messages is $\frac{120}{48} = 2.5$ times as large as the set of situations. Indeed, we can see some of that slop space in our algorithm: some hands are encoded by more than one message (any hand with more two cards of the same suit), and some messages never get used (any message which contains the card it encodes).

Generalize now to a deck with d cards, from which you draw a hand of n. Calculating as above, there are $d(d-1)\cdots(d-n+2)$ possible messages, and $\binom{d}{n}$ possible hands. The trick really is impossible (without subterfuge) if there are more hands than messages, i.e. unless $d \leq n! + n - 1$.

The remarkable theorem is that this upper bound on d is always attainable. While we calculated that there are enough messages to encode all the hands, it is far from obvious that we can match them up so each hand is encoded by a message using only the n cards available! But we can; the $n = 5$ trick, which we can do with 52 cards, can be done with a deck of 124. I will give an algorithm in a moment, but first an interesting nonconstructive proof.

The Birkhoff-von Neumann theorem states that the convex hull of the permutation matrices is precisely the set of *doubly stochastic matrices*: matrices with entries in $[0, 1]$ with each row and column summing to 1. We will use the equivalent discrete statement that any matrix of nonnegative integers with constant row and column sums can be written as a sum of permutation matrices [6]. To prove this by induction (on the constant sum) one need only show that any such matrix is entrywise greater than some permutation matrix. This is an application of Hall's Marriage theorem, which states that it is possible to arrange suitable marriages between n men and n women as long as any collection of k women can concoct a list of at least k men that someone among them considers an eligible

[3] Unpaid advertisement: for more information on this outstanding, intense, and enlightening introduction to mathematical thinking for talented high school students, contact David Kelly, Natural Science Department, Hampshire College, Amherst, MA 01002, or dkelly@hampshire.edu.

[4] For some reason I personally find it easier to encode and decode by scanning for the position of a given card: place the smallest card in the left/middle/right position to encode 12/34/56 respectively, placing medium before or after large to indicate the first or second number in each pair. The resulting order sml, slm, msl, lsm, mls, lms is just the lex order on the inverse of the permutation.

[5] If your goal is to confound instead, it is too transparent always to put the suit-indicating card first. Fitch recommended placing it (i mod 4)th for the ith performance to the same audience.

[6] Exercise: do so for your favorite magic square.

bachelor. To apply this to our nonnegative integer matrix, say that we can marry a row to a column only if their common entry is nonzero. The constant row and column sums ensure that any k rows have at least k columns they consider eligible.

Now consider the (very large) 0-1 matrix with rows indexed by the $\binom{d}{n}$ hands, columns indexed by the $d!/(d-n+1)!$ messages, and entries equal to 1 indicating that the cards used in the message all appear in the hand. When we take d to be our upper bound of $n!+n-1$, this is a square matrix, and has exactly $n!$ 1's in each row and column. We conclude that some subset of these 1's form a permutation matrix. But this is precisely a strategy for me and my lovely assistant - a bijection between hands and messages which can be used to represent them. Indeed, by the above paragraph, there is not just one strategy, but at least $n!$.

Perfection

Technically the above proof is constructive, in that the proof of Hall's Marriage theorem is itself a construction. But with $n = 5$ the above matrix has 225,150,024 rows and columns, so there is room for improvement. Moreover, we would like a workable strategy, one that we have a chance at performing without consulting a cheat sheet or scribbling on scrap paper. The perfect strategy below I learned from Elwyn Berlekamp, and I've been told that Stein Kulseth and Gadiel Seroussi came up with essentially the same one independently; likely others have done so too. Sadly, I have no information on whether Fitch Cheney thought about this generalization at all.

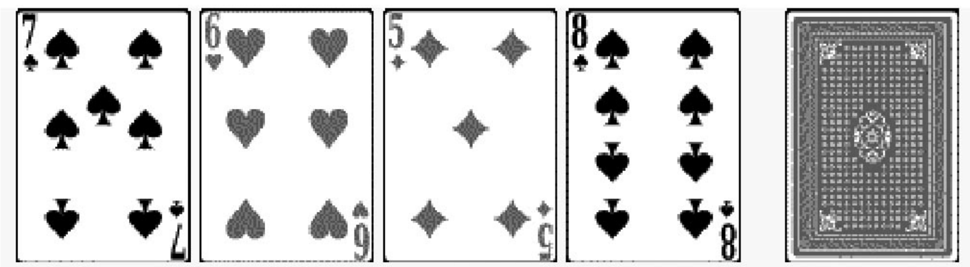
Suppose for simplicity of exposition that $n = 5$. Number the cards in the deck 0 through 123. Given a hand of five cards $c_0 < c_1 < c_2 < c_3 < c_4$, my assistant will choose c_i to remain hidden, where $i = c_0 + c_1 + c_2 + c_3 + c_4$ mod 5.

To see how this works, suppose the message consists of four cards which sum to s mod 5. Then the hidden card is congruent to $-s + i$ mod 5 if it is c_i. This is precisely the same as saying that if we renumber the cards from 0 to 119 by deleting the four cards used in the message, the hidden card's new number is congruent to $-s$ mod 5. Now it is clear that there are exactly 24 possibilities, and the permutation of the four displayed cards communicates a number p from 0 to 23, in "base factorial:" $p = d_1 1! + d_2 2! + d_3 3!$, where for lex order, $d_i \leq i$ counts how many cards to the right of the $n-i$th are smaller than it [7]. Decoding the hidden card is straightforward: let s be the sum of the four visible cards and calculate p as above based on their permutation, then take $5p+(-s$ mod 5) and carefully add 0, 1, 2, 3, or 4 to account for skipping the cards that appear in the message [8].

Having performed the 124-card version, I can report that with only a little practice it flows quite nicely. Berlekamp mentions that he has also performed the trick with a deck of only 64 cards, where the audience also flips a coin: after seeing four cards he both names the fifth and states whether the coin came up heads or tails. Encoding and decoding work just as before, only now when we delete the four cards used to transmit the message, the deck has 60 cards left, not 120, and the extra bit encodes the flip of the coin. If the 52-card version becomes too well known, I may need to resort to this variant to stay ahead of the crowd.

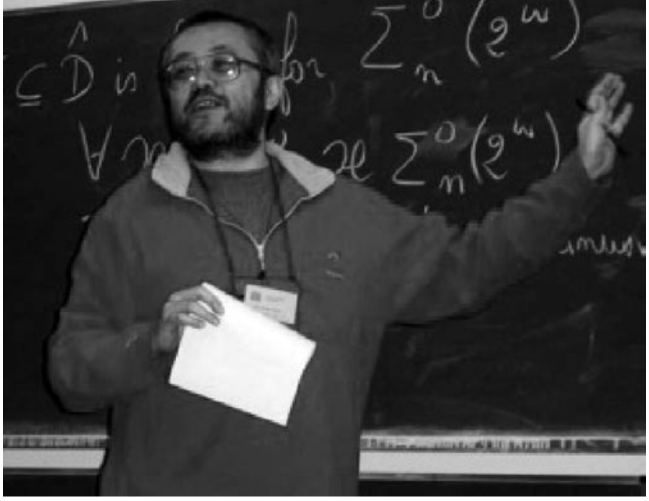
And finally a combinatorial question to which I have no answer: how many strategies exist? We probably ought to count equivalence classes modulo renumbering the underlying deck of cards. Perhaps we should also ignore composing a strategy with arbitrary permutations of the message - so two strategies are equivalent if, on every hand, they always choose the same card to remain hidden. Calculating the permanent of the aforementioned 225,150,024-row matrix seems like a bad way to begin. Is there a good one?

Acknowledgments: Much credit goes to Art Benjamin for popularizing the trick; I thank him, Persi Diaconis, and Bill Cheney for sharing what they knew of its history. In helping track Fitch Cheney from his Ph.D. through his mathematical career, I owe thanks to Marlene Manoff, Nora Murphy, Gregory Colati, Betsy Pittman, and Ethel Bacon, collection managers and archivists at MIT, MIT again, Tufts, Connecticut, and Hartford, respectively. Finally, you can't perform this trick alone. Thanks to my lovely assistants: Jessica Polito (my wife, who worked out the solution to the original trick with me on a long winter's walk), Benjamin Kleber, Tara Holm, Daniel Biss, and Sara Billey.



[7] Or, my preference (cf. footnote 4), di counts haw many cards larger than the ith smallest appear to the left of it. Either way, the conversion feels natural after practicing a few times.

[8] Exercise: verify that if your lovely assistant shows you the sequence of cards 37, 7, 94, 61 then the hidden card's number is a root of $x^3 - 18x^2 - 748x - 456$.



Alexander Shen:

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In this issue I present a collection of nice proofs that are based on some kind of a probabilistic argument, though the statement doesn't mention any probabilities. First a simple geometric example.

(1) It is known that ocean covers more than one half of the Earth's surface. Prove that there are two symmetric points covered by water.

Indeed, let X be a random point. Consider the events "X is covered by water" and "$-X$ is covered by water". (Here $-X$ denotes the point antipodal to X). Both events have probability more than 1/2, so they cannot be mutually exclusive.

Of course, the same (trivial) argument can be explained without any probabilities. Let $W \subset S^2$ be the subset of the sphere covered by water, and let $\mu(X)$ be the area of a region $X \subset S^2$. Then $\mu(W) + \mu(-W) > \mu(S^2)$, so $W \cap (-W) \neq \emptyset$.

However, as we see in the following examples, probability theory may be more than a convenient language to express the proof.

(2) A sphere is colored in two colors: 10% of its surface is white, the remaining part is black. Prove that there is a cube inscribed in the sphere such that all its 8 vertices are black.

Indeed, let us take a random cube inscribed in the sphere. For each vertex the probability of the event "vertex is white" is 0.1. Therefore the event "there exists a white vertex" has probability at most $8 \times 0.1 < 1$, therefore the cube has 8 black vertices with a positive probability.

This argument assumes implicitly that there exists a random variable (on some sample space) whose values are cubes with numbered vertices and each vertex is uniformly distributed over the sphere. The easiest way to construct such a variable is to consider $SO(3)$ with an invariant measure as a sample space. It seems that here probability language is more important: if we did not have probabilities in mind, why should we consider an invariant measure on $SO(3)$?

Now let us switch from toy examples to more serious ones.

(3) In this example we want to construct a bipartite graph with the following properties:
(a) both parts L and R (called "left" and "right") contain n vertices;
(b) each vertex on the left is connected to at most eight vertices on the right;
(c) for each set $X \subset L$ that contains at least $0.5n$ vertices the set of all neighbors of all vertices in X contains at least $0.7n$ vertices.

(These requirements are taken from the definition of "expander graphs"; constants are chosen to simplify calculations.)

We want to prove that for each n there exists a graph that satisfies conditions (a) - (c). For small n it is easy to draw such a graph (e.g., for $n \leq 8$ we just connect all the vertices in L and in R), but it seems that in the general case there is no simple construction with an easy proof.

However, the following probabilistic argument proves that such graphs do exist. For each left vertex x pick eight random vertices on the right (some of them may coincide) and call these vertices neighbors of x. All choices are independent. We get a graph that satisfies (a) and (b); let us prove that it satisfies (c) with positive probability. Fix some $X \subset L$ that has at least $0.5n$ vertices and some $Y \subset R$ that has less than $0.7n$ vertices. What is the probability of the event "All neighbors of all elements of X belong to Y"? For each fixed x is an element of X the probability that all eight random choices produce an element from Y, does not exceed $(0.7)^8$. For different elements of X choices are independent, so the resulting probability is bounded by $(0.7^8)^{0.5n} = 0.7^{4n}$. There are fewer than $2n$ different possibilities for each of the sets X and Y, so the probability of the event "there exist X and Y such that $|X| \geq 0.5n$, $|Y| < 0.7n$, and all neighbors of all vertices in X belong to Y" does not exceed $2n \times 2n \times 0.74n = 0.982n < 1$. This event embodies the negation of the requirement (c), so we are done.

All the examples above follow the same scheme. We want to prove that an object with some property α exists. We consider a suitable probability distribution and prove that a random object has property α with nonzero probability. Let us consider now two examples of a more general scheme: if the expectation of a random variable ζ is greater than some number λ, some values of ζ are greater than λ.

(4) A piece of paper has area 10 square centimeters. Prove that

Probabilistic proofs

it can be placed on the integer grid (the side of whose square is 1 cm) so that at least 10 grid points are covered.

Indeed, let us place a piece of paper on the grid randomly. The expected number of grid points covered by it is proportional to its area (because this expectation is an additive function). Moreover, for big pieces the boundary effects are negligible, and the number of covered points is close to the area (relative error is small). So the coefficient is 1, and the expected number of covered points is equal to the area. If the area is 10, the expected number is 10, so there must be at least one position where the number of covered points is 10 or more.

(5) A stone is convex; its surface has area S. Prove that the stone can be placed in the sunlight in such a way that the shadow will have area at least $S/4$. (We assume that light is perpendicular to the plane where the shadow is cast; if it is not, the shadow only becomes bigger.)

Let us compute the expected area of the shadow. Each piece of the surface contributes to the shadow exactly twice (here convexity is used), so the shadow is half the sum of the shadows of all pieces. Taking into account that for each piece all possible directions of light are equiprobable, we see that the expected area of the shadow is proportional to the area of the stone surface. To find the coefficient, take the sphere as an example: it has area $4\pi r^2$ and its shadow has area πr^2, so the expected shadow area is $S/4$.

(6) We finish our collection of nice probabilistic proofs with a well-known example, so nice and unexpected that it cannot be omitted. It is the probabilistic proof of the Weierstrass theorem saying that *any continuous function can be approximated by a polynomial.* (As far as I know, this proof is due to S.N. Bernstein.)

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Let p be a real number in $[0, 1]$. Construct a random variable in the following way. Make n independent trials, the probability of success in each of them being p. If the number of successes is k, take $f(k/n)$ as the value of the random variable. For each p we get a random variable. Its expectation is a function of p; let us call it $f_n(p)$.

It is easy to see that for each n the function f_n is a polynomial. (What else can we get if the construction uses only a finite number of f-values?) On the other hand, f_n is close to f, because for any p the ratio k/n is close to p with overwhelming probability (assuming n is big enough); so in most cases the value of $f(k/n)$ is close to $f(p)$, since $f(p)$ is uniformly continuous. The formal argument requires some estimates of probabilities (Chernoff bound or whatever), but we omit the details.

Alexander Shen:

Lights out

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The board for this game is an $m \times n$ rectangular array of lamps. Each lamp may be on or off. Each lamp works as a button changing the state (on/off) of the lamp and all its neighbors. Thus the maximal number of lamps affected by one button is five, the minimal number is three (for the corner button). Initially all lamps are on; the goal is to switch all the lamps off by a succession of button-pushes.

I heard about this game about ten years ago from Michael Sipser (MIT), who told me that it is always solvable and there is a very nice proof of this using linear algebra. Recently Prof. Oscar Martin-Sanches and Cristobal Pareja-Flores wrote an article about this puzzle (to appear; see also their site http://dalila.sip.ucm.es/miembros/cpareja/lo), where they provide a detailed proof for the 5×5-game. (By the way, they have found this puzzle in toy stores!)

Here is the solution using linear algebra. First of all, we may forget about the rectangle; let V be the set of vertices of an arbitrary undirected graph. Each vertex has a lamp and a button that changes the state of this lamp and all its neighbors. The set of all configurations of lamps forms a linear space over $\mathbb{Z}/2\mathbb{Z}$. Each vector is a function of type $V \to \{0,1\}$. Here 1/0 means on/off, and vector addition is performed modulo 2. The dimension of this space is the number of lamps, i.e., $|V|$. For each vertex v we consider a function f_v that equals 1 in the neighborhood of v and 0 elsewhere. We need to prove that the function u that is equal to 1 everywhere can be represented as a linear combination of functions f_v.

It is enough to show that any linear functional a that maps all f_v to zero equals zero on u. Any linear functional $\alpha : \{0,1\}^V \to \{0,1\}$ can be represented as $\alpha(f) = \Sigma\{f(v)|v \ \ is\ an\ element\ of\ A\}$ for some $A \subseteq V$ (the sum is computed modulo 2). Therefore the statement can be reformulated as follows: if A has even-sized intersection with the neighborhood of any vertex v, then $|A|$ is even.

To see that this inference holds, consider the restriction of our graph to A. Each vertex a is an element of A has odd degree in the restricted graph, but the sum of the degrees of graph A is of course even; therefore the number of vertices of the restricted graph, $|A|$, is even.

We get also the criterion saying whether the state $c \in \{0,1\}^V$ is solvable. Here it is: $\Sigma\{c(v)|v\ is\ an\ element\ of A\} = 0$ for any subset $A \subseteq V$ having even-size intersection with the neighborhood of any vertex. (Such a set A can be called "neutral": if we press all buttons in A, all lamps return to their initial state.) One may ask for an "elementary" solution; indeed Sipser reports,

...An epilog to the lamp problem. A generalization (which may make the problem easier) appeared in the problem section of American Mathematical Monthly [see problem 10197, vol. 99, no. 2, February 1992, p. 162 and vol. 100, no. 8, Oct. 93, pp. 806-807]. A very sharp new student here (named Marcos Kiwi) found a nice solution to it. Say that the lamps are nodes of a given undirected graph. The button associated with a lamp switches both its state and the state of all its neighbors. Then we prove that there is a way to switch all states as follows. Use induction on n (the number of nodes of the graph). First say n is even. For each lamp, remove it, and take the inductively given solution on the smaller graph. Replace the lamp and see whether the solution switches it. If yes, then we are done. If no for every lamp, then take the linear sum of all the above solutions given for all the lamps. Every lamp is switched an odd number of times (n - 1), so we are done. If n is odd, then there must be a node of even degree. Do the above procedure for only the nodes in the neighborhood of this node, including itself. In addition press the button of this lamp. This also switches all lamps an odd number of times.

Last year this problem appeared on the All-Russia Math Olympiad. One of the participants, Ilia Meszirov, rediscovered Kiwi's argument. He also gave an elementary proof (not using linear algebra) for the statement mentioned above (a state having even-sized intersection with any neutral set is solvable).

Pentangram—A New Puzzle

Klaus Kühnle

This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

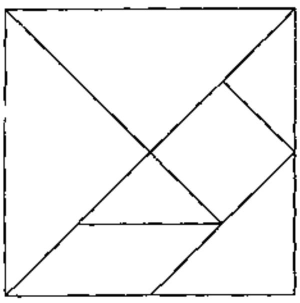
We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

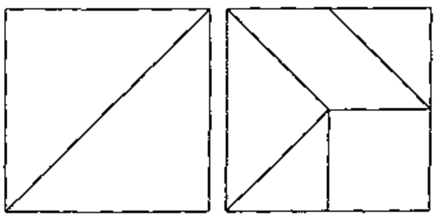
Dette er en direkte gengivelse af en artikel fra Mathematical Intelligencer, vol 21, nummer 2, 1999.

Please send all submissions to the Mathematical Entertainments Editor, **Alexander Shen,** Institute for Problems of Information Transmission, Ermolovoi 19, K-51 Moscow GSP-4, 101447 Russia; e-mail:shen@landau.ac.ru

The Classic Tangram

The well-known Chinese tangram is a puzzle consisting of seven pieces that can be arranged into either one square of area 2 or two squares of area 1 each:





Although there are loads of other more or less funny shapes that can be built from those seven pieces, these two reveal best the immanent essence of the puzzle from the mathematical point of view: The tangram is based on $\sqrt{2}$ as some sort of magic number. The ratio between any two lengths occurring as side-lengths of the seven pieces is some power of $\sqrt{2}$.

The right-angled, isosceles triangle occurs in three different sizes as pieces of the puzzle. The ratio between the hypotenuse and the legs of such a triangle is $\sqrt{2}$. Furthermore, the hypotenuse of a small triangle is just as long as the legs of the next larger one; in other words, the next larger triangle is scaled by a factor of $\sqrt{2}$. As a consequence, its area is just doubled.

The same relation holds for the arranged squares shown above. The side-length of the big square is equal to the diagonals of the small squares, which are just $\sqrt{2}$ times their side-lengths.

Despite the triviality of all that has been said, it can help the solver of the puzzle. Since $\sqrt{2}$ is irrational, a length that is the sum of a non-zero integer and a non-zero integer multiple of $\sqrt{2}$ can never be an integer or an integer multiple of $\sqrt{2}$. This implies that it is predetermined how the pieces may be rotated in order to be useful. Thus in the above illustration, all pieces in the small squares are rotated by an odd multiple of $\frac{\pi}{4}$ in comparison with their appearance in the big square. And the preceding argument implies that this is necessary.

Variants of the Same Idea

The classic tangram is based on $\sqrt{2}$, the ratio between the side-length and the diagonal of a square. The angles occurring are all multiples of the angle between a side and a diagonal, namely $\frac{\pi}{4}$. Equally well one could design a similar puzzle where the pieces have to be arranged to regular hexagons, and the magic number would be $\sqrt{3}$, the length of the chord in a hexagon of side-length 1. (The other chord, the diagonal, has length 2; and an integer magic number would not make for a very interesting puzzle.) Such a puzzle, based on $\sqrt{3}$, could be arranged to one big hexagon or to three small ones, where the chord-lengths of the small hexagons would be equal to the side-length of the big one. Here, all angles occurring should be multiples of $\frac{\pi}{6}$. The sides whose lengths are even/odd powers of

$\sqrt{3}$ would have to be placed in a rotation which is an even/odd multiple of $\frac{\pi}{6}$.

Another variant would be the arrangement of equilateral triangles with $\frac{\sqrt{3}}{2}$, which is the ratio between the height and the side-length, as the magic number. In this case, three big triangles would have the same area as four small ones if one stuck to the principle that they be scaled by the magic number. The angles should again be multiples of the smallest angle between the two straight lines whose ratios of lengths define the magic number, namely $\frac{\pi}{6}$.

I do not attempt to give a comprehensive survey; let me just mention that for regular polygons with (say) 7, 9, 11, 13, or 19 sides, the ratios between chords and sides are transcendental numbers, so I do not see any way to design a puzzle in the same fashion from them.

The variant that attracted me most is the regular pentagon, whose ratio between chord-length and side-length veritably is a magic number, namely the golden ratio.

The Golden Ratio and its Square

The golden ratio is ubiquitous in all sorts of mathematics; I will resist the temptation of expatiating about it. Let us just fix upon φ as a brief name for it: $\varphi = \frac{\sqrt{5}+1}{2}$.

As in the other variants, the puzzle shall be such that its pieces can be arranged into a number of small pentagons as well as into a number of big pentagons. But this time the square of our magic number is irrational; hence, for all integers n and m, n pentagons of a certain size will occupy an area different from that of m pentagons whose sizes are scaled by φ. Instead, we will have to take pentagons of three different sizes with a scaling factor of φ between them. The areas of these pentagons are then 1, φ^2, and φ^4 respectively, and three medium-sized pentagons have the same area as a small and a big one together:

$$3\varphi^2 = 1 + \varphi^4.$$

How did I hit on this simple equality?

All that can be said about the golden ratio boils down to the simple equation

$$\varphi^2 - \varphi - 1 = 0,$$

which is usually used as a definition for φ. We are interested here in descriptions of φ^2; so, we just adjoin

$$\varphi^2 = A$$

and eliminate φ from the system of two equations. The result is

$$A^2 - 3A + 1 = 0,$$

which is the simplest possible statement about A. Consequently, the above requirement that a small and a big pentagon together occupy the same area as three medium-sized ones is the simplest I could have made.

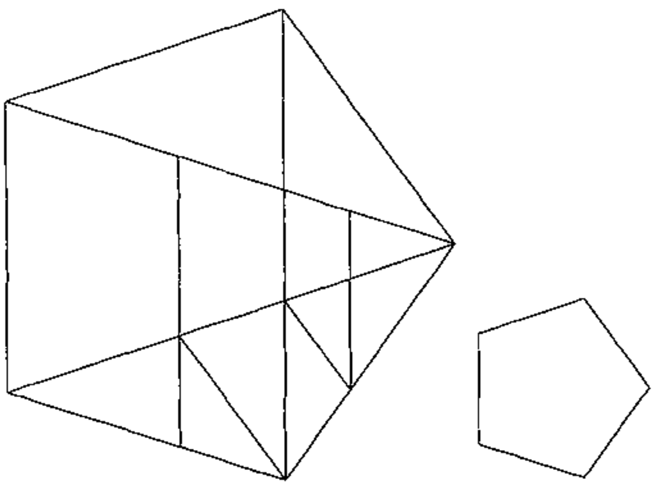
The smallest angle between a side and a chord in the regular pentagon is $\frac{\pi}{5}$, hence all angles occurring shall be multiples of $\frac{\pi}{5}$. In analogy to other cases, one is tempted to conjecture that a side having a length that is an odd power of φ has to occur in a rotation that is an odd multiple of $\frac{\pi}{5}$. But since $\forall n \in \mathbb{Z}$: $\varphi^n + \varphi^{n+1} = \varphi^{n+2}$ and any power of φ apart from $\varphi^0 = 1$ is a sum of a nonzero integer multiple of $\frac{1}{2}$ and a nonzero integer multiple of $\frac{\sqrt{5}}{2}$, such a statement can in this case not be made. This, at least in my opinion, adds to the spirit of symmetry accounting for the fascination of such a puzzle.

Another consequence of $\forall n \in \mathbb{Z}$: $\varphi^n + \varphi^{n+1} = \varphi^{n+2}$ is that, theoretically, any power of φ as a goal length can be reached by cumulating smaller powers of φ regardless of how (i.e., with what powers of φ) one has started to approach this goal. Of course, this is not true in practice, because no arbitrarily small powers of φ occur as side-lengths of pieces in the puzzle; but still, this principle will have its bearing.

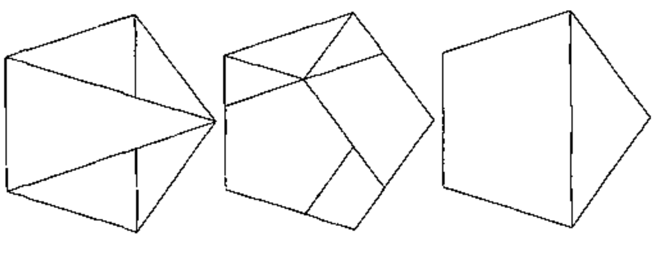
The Details of the Pentangram

These principal features, arising from properties of the pentagon and the golden ratio, do not at all determine the details. Of the many possibilities of cutting a small and a big pentagon into pieces that can be rearranged into three medium-sized pentagons, I rather unpremeditatedly hit on one consisting of thirteen pieces of four different shapes in up to three different magnifications, where, of course, the magnification ratio is φ.

Arrangement of a big and a small pentagon:



Arrangement of three medium-sized pentagons (with the same pieces):



The two different shapes of triangles are the only possible ones given the obligation that all angles be multiples of $\frac{\pi}{5}$; the pentagon is a reminiscence of the shape that has to be arranged; and the quadrilateral is just one of the many alternatives with angles multiples of $\frac{\pi}{5}$ and side-lengths powers of φ.

As in the classic tangram, this division is not the one with the least number of pieces necessary to obtain the two principal shapes, because such a minimal division would probably ease the task of solving the puzzle; moreover it might decrease the number of other shapes that can be built. A small collection of such other shapes is given in the appendix; the reader is invited to discover more of them.

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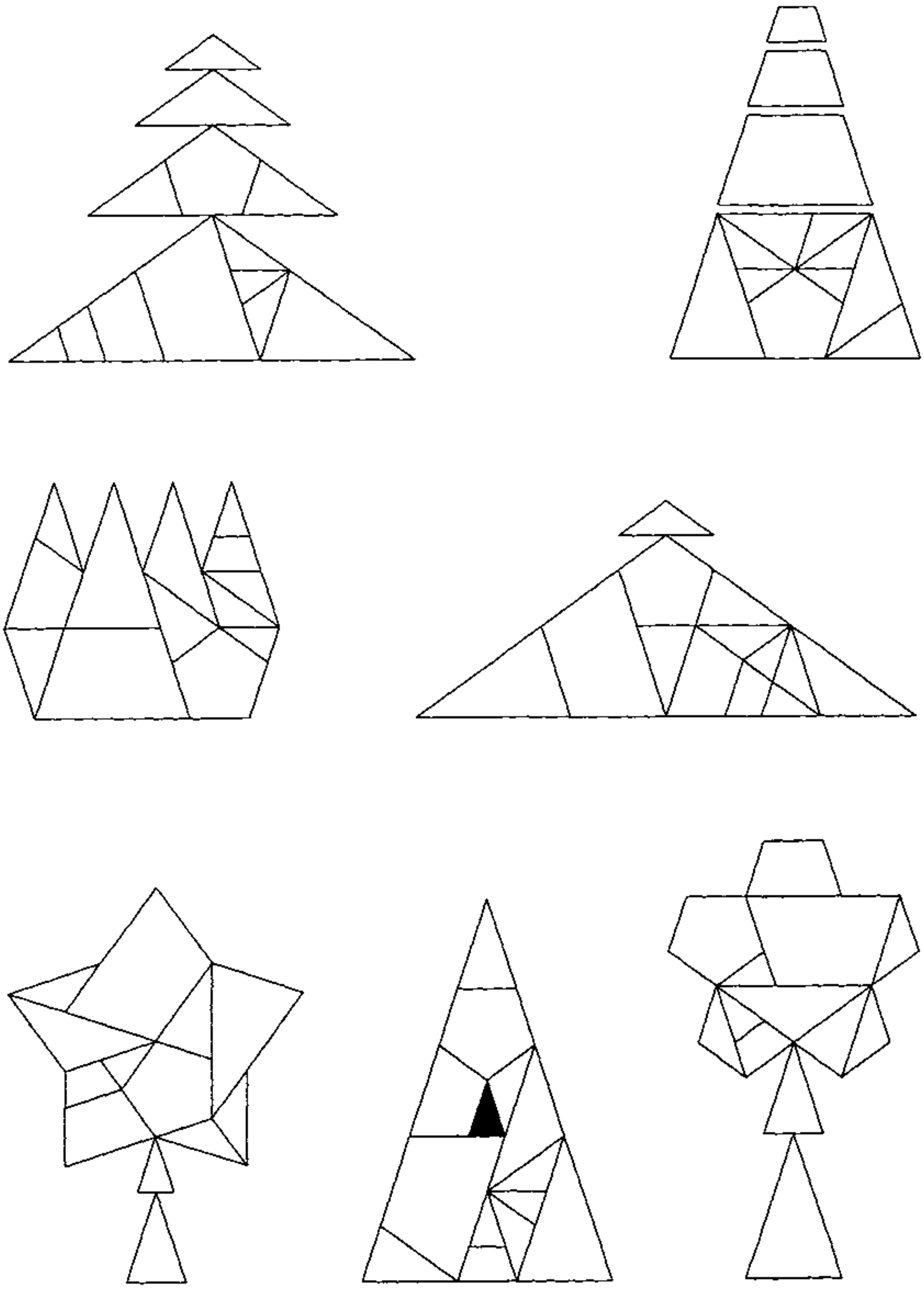
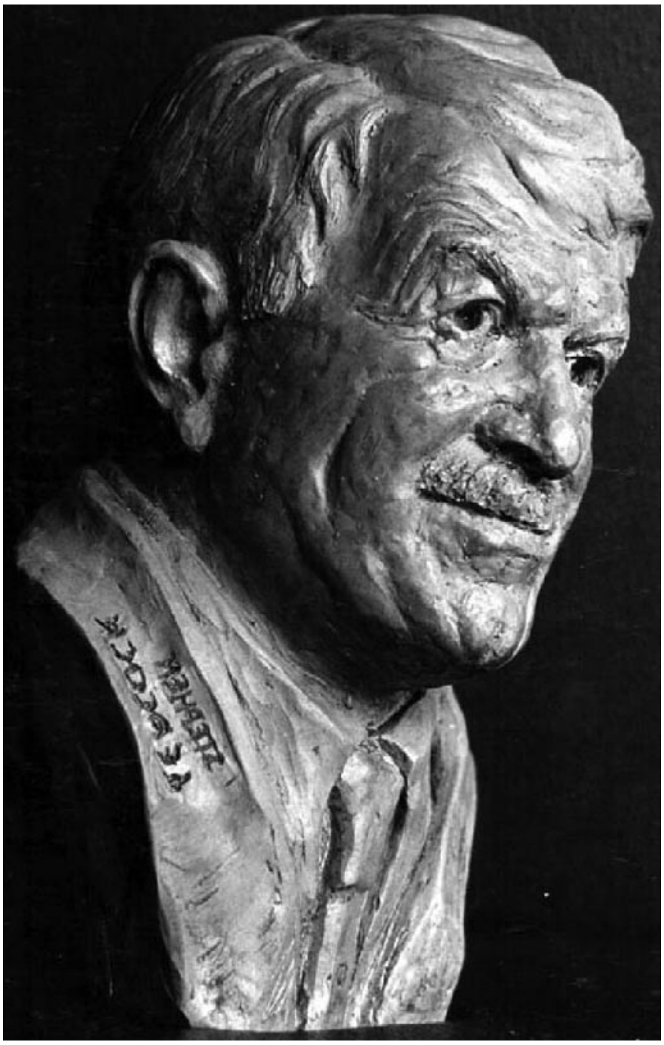
Appendix: A Small Collection of Other Arrangements



Stephen Leacock



Stephen Butler Leacock (1869-1944) er en kendt canadisk humorist. I hans bog "Literary lapses" fra 1910 findes et par noveller, som viser, at han må have haft kendskab til matematik. Det er måske, fordi han før bogens udgivelse havde opnået en PhD i økonomi og politisk videnskab fra University of Chicago under vejledning af Thorstein Veblen. Gad vide, om Leacocks forhold til matematik har været specielt harmonisk.

To noveller

A, B, and C

THE HUMAN ELEMENT IN MATHEMATICS

The student of arithmetic who has mastered the first four rules of his art, and successfully striven with money sums and fractions, finds himself confronted by an unbroken expanse of questions known as problems. These are short stories of adventure and industry with the end omitted, and though betraying a strong family resemblance, are not without a certain element of romance.

The characters in the plot of a problem are three people called A, B, and C. The form of the question is generally of this sort:

"A, B, and C do a certain piece of work. A can do as much work in one hour as B in two, or C in four. Find how long they work at it."

Or thus:

"A, B, and C are employed to dig a ditch. A can dig as much in one hour as B can dig in two, and B can dig twice as fast as C. Find how long, etc. etc."

Or after this wise:

"A lays a wager that he can walk faster than B or C. A can walk half as fast again as B, and C is only an indifferent walker. Find how far, and so forth."

The occupations of A, B, and C are many and varied. In the older arithmetics they contented themselves with doing "a certain piece of work." This statement of the case however, was found too sly and mysterious, or possibly lacking in romantic charm. It became the fashion to define the job more clearly and to set them at walking matches, ditch-digging, regattas, and piling cord wood. At times, they became commercial and entered into partnership, having with their old mystery a "certain" capital. Above all they revel in motion. When they tire of walking-matches--A rides on horseback, or borrows a bicycle and competes with his weaker-minded associates on foot. Now they race on locomotives; now they row; or again they become historical and engage stage-coaches; or at times they are aquatic and swim. If their occupation is actual work they prefer to pump water into cisterns, two of which leak through holes in the bottom and one of which is water-tight. A, of course, has the good one; he also takes the bicycle, and the best locomotive, and the right of swimming with the current. Whatever they do they put money on it, being all three sports. A always wins.

In the early chapters of the arithmetic, their identity is concealed under the names John, William, and Henry, and they wrangle over the division of marbles. In algebra they are often called X, Y, Z. But these are only their Christian names, and they are really the same people.

Now to one who has followed the history of these men through countless pages of problems, watched them in their leisure hours dallying with cord wood, and seen their panting sides heave in the full frenzy of filling a cistern with a leak in it, they become something more than mere symbols. They appear as creatures of flesh and blood, living men with their own passions, ambitions, and aspirations like the rest of us. Let us view them in turn. A is a full-blooded blustering fellow, of energetic temperament, hot-headed and strong-willed. It is he who proposes everything, challenges B to work, makes the bets, and bends the others to his will. He is a man of great physical strength and phenomenal endurance. He has been known to walk forty-eight hours at a stretch, and to pump ninety-six. His life is arduous and full of peril. A mistake in the working of a sum may keep him digging a fortnight without sleep. A repeating decimal in the answer might kill him.

B is a quiet, easy-going fellow, afraid of A and bullied by him, but very gentle and brotherly to little C, the weakling. He is quite in A's power, having lost all his money in bets.

Poor C is an undersized, frail man, with a plaintive face. Constant walking, digging, and pumping has broken his health and ruined his nervous system. His joyless life has driven him to drink and smoke more than is good for him, and his hand often shakes as he digs ditches. He has not the strength to work as the others can, in fact, as Hamlin Smith has said, "A can do more work in one hour than C in four."

The first time that ever I saw these men was one evening after a regatta. They had all been rowing in it, and it had transpired that A could row as much in one hour as B in two, or C in four. B and C had come in dead fagged and C was coughing badly. “Never mind, old fellow,” I heard B say, “I’ll fix you up on the sofa and get you some hot tea.” Just then A came blustering in and shouted, “I say, you fellows, Hamlin Smith has shown me three cisterns in his garden and he says we can pump them until to-morrow night. I bet I can beat you both. Come on. You can pump in your rowing things, you know. Your cistern leaks a little, I think, C.” I heard B growl that it was a dirty shame and that C was used up now, but they went, and presently I could tell from the sound of the water that A was pumping four times as fast as C.

For years after that I used to see them constantly about town and always busy. I never heard of any of them eating or sleeping. Then owing to a long absence from home, I lost sight of them. On my return I was surprised to no longer find A, B, and C at their accustomed tasks; on inquiry I heard that work in this line was now done by N, M, and O, and that some people were employing for algebraica jobs four foreigners called Alpha, Beta, Gamma, and Delta.

Now it chanced one day that I stumbled upon old D, in the little garden in front of his cottage, hoeing in the sun. D is an aged labouring man who used occasionally to be called in to help A, B, and C. “Did I know ‘em, sir?” he answered, “why, I knowed ‘em ever since they was little fellows in brackets. Master A, he were a fine lad, sir, though I always said, give me Master B for kind-heartedness-like. Many’s the job as we’ve been on together, sir, though I never did no racing nor aught of that, but just the plain labour, as you might say. I’m getting a bit too old and stiff for it nowadays, sir--just scratch about in the garden here and grow a bit of a logarithm, or raise a common denominator or two. But Mr. Euclid he use me still for them propositions, he do.”

From the garrulous old man I learned the melancholy end of my former acquaintances. Soon after I left town, he told me, C had been taken ill. It seems that A and B had been rowing on the river for a wager, and C had been running on the bank and then sat in a draught. Of course the bank had refused the draught and C was taken ill. A and B came home and found C lying helpless in bed. A shook him roughly and said, “Get up, C, we’re going to pile wood.” C looked so worn and pitiful that B said, “Look here, A, I won’t stand this, he isn’t fit to pile wood to-night.” C smiled feebly and said, “Perhaps I might pile a little if I sat up in bed.” Then B, thoroughly alarmed, said, “See here, A, I’m going to fetch a doctor; he’s dying.” A flared up and answered, “You’ve no money to fetch a doctor.” “I’ll reduce him to his lowest terms,” B said firmly, “that’ll fetch him.” C’s life might even then have been saved but they made a mistake about the medicine. It stood at the head of the bed on a bracket, and the nurse accidentally removed it from the bracket without changing the sign. After the fatal blunder C seems to have sunk rapidly. On the evening of the next day, as the shadows deepened in the little room, it was clear to all that the end was near. I think that even A was affected at the last as he stood with bowed head, aimlessly offering to bet with the doctor on C’s laboured breathing. “A,” whispered C, “I think I’m going fast.” “How fast do you think you’ll go, old man?” murmured A. “I don’t know,” said C, “but I’m going at any rate.”--The end came soon after that. C rallied for a moment and asked for a certain piece of work that he had left downstairs. A put it in his arms and he expired. As his soul sped heavenward A watched its flight with melancholy admiration. B burst into a passionate flood of tears and sobbed, “Put away his little cistern and the rowing clothes he used to wear, I feel as if I could hardly ever dig again.”--The funeral was plain and unostentatious. It differed in nothing from the ordinary, except that out of deference to sporting men and mathematicians, A engaged two hearses. Both vehicles started at the same time, B driving the one which bore the sable parallelopiped containing the last remains of his ill-fated friend. A on the box of the empty hearse generously consented to a handicap of a hundred yards, but arrived first at the cemetery by driving four times as fast as B. (Find the distance to the cemetery.) As the sarcophagus was lowered, the grave was surrounded by the broken figures of the first book of Euclid.--It was noticed that after the death of C, A became a changed man. He lost interest in racing with B, and dug but languidly. He finally gave up his work and settled down to live on the interest of his bets.--B never recovered from the shock of C’s death; his grief preyed upon his intellect and it became deranged. He grew moody and spoke only in monosyllables. His disease became rapidly aggravated, and he presently spoke only in words whose spelling was regular and which presented no difficulty to the beginner. Realizing his precarious condition he voluntarily submitted to be incarcerated in an asylum, where he abjured mathematics and devoted himself to writing the History of the Swiss Family Robinson in words of one syllable.

"As the sarcophagus was lowered, the grave was surrounded by the broken figures of the first book of Euclid"

Boarding-House Geometry

DEFINITIONS AND AXIOMS

All boarding-houses are the same boarding-house.

Boarders in the same boarding-house and on the same flat are equal to one another.

A single room is that which has no parts and no magnitude.

The landlady of a boarding-house is a parallelogram--that is, an oblong angular figure, which cannot be described, but which is equal to anything.

A wrangle is the disinclination of two boarders to each other that meet together but are not in the same line.

All the other rooms being taken, a single room is said to be a double room.

POSTULATES AND PROPOSITIONS

A pie may be produced any number of times. The landlady can be reduced to her lowest terms by a series of propositions.

A bee line may be made from any boarding-house to any other boarding-house.

The clothes of a boarding-house bed, though produced ever so far both ways, will not meet.

Any two meals at a boarding-house are together less than two square meals.

If from the opposite ends of a boarding-house a line be drawn passing through all the rooms in turn, then the stovepipe which warms the boarders will lie within that line.

On the same bill and on the same side of it there should not be two charges for the same thing.

If there be two boarders on the same flat, and the amount of side of the one be equal to the amount of side of the other, each to each, and the wrangle between one boarder and the landlady be equal to the wrangle between the landlady and the other, then shall the weekly bills of the two boarders be equal also, each to each.

For if not, let one bill be the greater. Then the other bill is less than it might have been--which is absurd.

Interview with the winners of the Abel Prize 2008:

John G. Thompson and Jacques Tits

Martin Raussen (Aalborg, Denmark) and Christian Skau (Trondheim, Norway)
Oslo, May 19, 2008

(Denne artikel udkom først i EMS Newsletter 69, September 2008, s. 31-38. Vi takker for tilladelsen til at genoptrykke den.)

Early experiences

Raussen & Skau: On behalf of the Norwegian, Danish and European Mathematical Societies we want to congratulate you for having been selected as Abel Prize winners for 2008. In our first question we would like to ask you when you first got interested in mathematics:

Were there any mathematical results or theorems that made a special impression on you in your childhood or early youth? Did you make any mathematical discoveries during that time that you still remember?

Tits: I learned the rudiments of arithmetic very early; I was able to count as a small child; less than four years, I believe. At the age of thirteen, I was reading mathematical books that I found in my father's library and shortly after, I started tutoring youngsters five years older than me who were preparing the entrance examination at the École Polytechnique in Brussels. That is my first recollection.

At that time I was interested in analysis but later on, I became a geometer. Concerning my work in those early years, I certainly cannot talk about great discoveries, but I think that some of the results I obtained then are not without interest.

My starting subject in mathematical research has been the study of strictly triple transitive groups; that was the subject essentially given to me by my professor. The problem was this: We knew axiomatic projective geometry in dimension greater than one. For the one-dimensional case, nobody had given an axiomatic definition. The one-dimensional case corresponds to PSL(2). My teacher gave me the problem of formulating axiomatics for these groups. The idea was to take triple transitivity as the first axiom. So I started by this kind of problem: giving axiomatics of projective geometry based on triple transitivity. Of course, I was then led naturally to consider quadruple and quintuple transitivity. That is how I rediscovered all the Mathieu groups, except, strangely enough, the biggest one, the quintuple transitive. I had to rediscover that one in the literature!

So you didn't know about the Mathieu groups when you did this work?

Tits: No, I didn't.

How old were you at that time?

Tits: 18 years old, I suppose. In fact, I first found all strictly quadruple transitive groups. They were actually known by Camille Jordan. But I didn't know the work of Camille Jordan at the time. I rediscovered that.

You must have been much younger than your fellow students at the time. Was it a problem to adjust in an environment where you were the youngest by far?

Tits: I am very grateful to my fellow students and also to my family. Because I was what is sometimes called a little genius. I was much quicker than all the others. But nobody picked up on that, they just let it go. My father was a little bit afraid that I would go too fast. My mother knew that this was exceptional, but she never boasted about it. In fact, a female neighbour said to my mother: "If I had a son like that, I would go around and boast about it." My mother found that silly. I was not at all put on a pedestal.

Hardy once said that mathematics is a young man's game. Do you agree?

Tits: I think that it is true to a certain extent. But there are people who do very deep things at a later age. After all, Chevalley's most important work was done when he was more than 40 years old and even perhaps later. It is not an absolute rule. People like to state such rules. I don't like them really.

Thompson: Well, it is true that you don't have any childhood geniuses in politics. But in chess, music and mathematics, there is room for childhood exceptionalism to come forth. This is certainly very obvious in the case of music and chess and to some extent in mathematics. That might sort of skew the books in a certain direction.

As far as Hardy's remark is concerned I don't know what he was feeling about himself at the time he made that remark. It could be a way for person to say: "I am checking out now, I reached the age where I don't want to carry on." I don't know what the sociologists and psychologists say; I leave it to them. I have seen mathematicians reach the age of 50 and still be incredible lively. I don't see it as a hard and fast rule. But then Tits and I are really in no position to talk given our age.

John von Neumann said, exaggerating a little, that whatever you do in mathematics beyond 30 is not worth anything, at least not compared to what you had done before 30. But when he himself reached the age of 30, he pushed this limit. Experience comes in etc…

Thompson: But he was a prodigy. So he knows the childhood side of it.

Tits: We all have known very young and bright mathematicians. The point is that to find deep mathematics, it is not necessary to have all the techniques. They can find results that are deep without having all of those techniques at hand.

What about your memories on early mathematical experiences, Professor Thompson?

Thompson: I don't have any particularly strong memories. I have an older brother, three years older than me, who was very good at math. He was instrumental, I guess, in interesting me in very elementary things. He was obviously more advanced than I was.

We also played cards in our family. I liked the combinatorics in card play. At that time, I was 10 or 12 years old. I also liked playing chess. I never got any good at it but I liked it. When my brother went to the university, he learned about calculus and he tried to explain it to me. I found it totally incomprehensible, but it intrigued me, though. I did get books out of the library myself. But I didn't make too much progress without him.

Early group theory

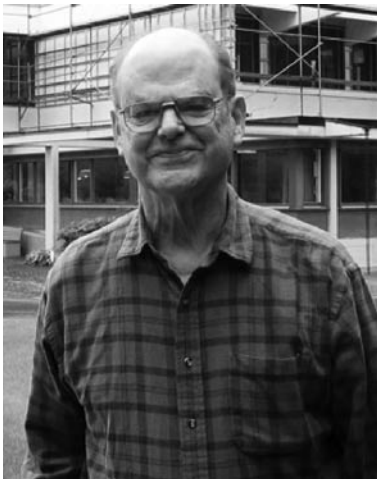
You have received this year's Abel Prize for your achievements in group theory. Can we start with a short historical introduction to the subject? We would like to ask you to tell us how the notion of a group came up and how it was developed during the 19th century. In fact, Norwegian mathematicians played quite an important role in that game, didn't they?

Tits: Well, when you talk about groups it is natural to talk about Galois. I think Abel did not use groups in his theory — do you know?

Thompson: At least implicitly. I think the equation of the fifth degree comes in there. It was a great eye opener). I myself looked at a very well-known paper of Lagrange, I think around 1770, before the French revolution. He examined equations and he also said something about equations of degree five. He was definitely getting close to the notion of a group. I don't know about the actual formal definition. I guess we have to attribute it to Galois. Anyway, it was certainly he that came up with the notion of a normal subgroup, I am pretty sure that was Galois' idea. He came up with the idea of a normal subgroup which is really basic.

Tits: But the theorem on the equation of degree five was discovered first by Abel, I think. Of course Galois had a technique which helped with many equations of different types that Abel did not have. Galois was really basically an algebraist, whereas Abel was also an analyst. When we now talk about abelian functions: these ideas go back to Abel.

Can you explain why simple groups are so important for the classification of finite groups in general? That realization came about, we guess, with Camille Jordan and his decomposition theorem. Is that correct?



John G. Thompson

Tits: You see, I think that one of the dreams of these people was always to describe all groups. And if you want to describe all groups you decompose them. The factors are then simple. I think that was one of the aims of what they were doing. But of course they didn't go that far. It is only very recently that one could find all finite simple groups, a solution to the problem to which John Thompson contributed in a major way.

What about Sylow and Lie in the beginning of group theory?

Thompson: Those are two other Norwegians.

Tits: Lie played an important role in my career. In fact, practically from the beginning, the main subject of my work has centred around the so-called exceptional Lie groups. So the work of Lie is basic in what I have done.

Could you comment on the work of Frobenius and Burnside?

Thompson: Of course. After the last half of the 19th century Frobenius among other things put the theory of group characters on a solid basis. He proved the orthogonality relations and talked about the transfer map. Burnside eventually got on the wagon there. And eventually he proved his p^aq^b-theorem, the two prime theorem, using character theory, namely that groups of these orders are solvable. That was a very nice step forward, I feel. It showed the power of character theory which Frobenius had already done. Frobenius also studied the character theory of the symmetric groups and multiply transitive permutation groups. I don't know how much he thought of the Mathieu groups. But they were pretty curious objects that had been discovered before character theory. For a while there was quite a bit of interest in multiply transitive permutation groups: quite complicated combinatorial arguments. Burnside and Frobenius were very much in the thick of things at that stage.

Tits: When I was a young mathematician. I was very ignorant of the literature. For instance, I rediscovered a lot of the results that were known about multiply transitive groups; in particular, on the strictly 4-fold and 5-fold transitive groups. Fortunately, I did this with other methods than the ones that were used before. So these results were in fact new in a certain sense.

Was it a disappointment to discover that these results had been discovered earlier?

Tits: Not too much.

Burnside was also interesting because he posed problems and conjectures that you and others worked on later, right?

Thompson: Right, well I sort of got started on working on the Frobenius conjecture, which was still open. I think it was Reinhold Baer or maybe Marshall Hall who told me about the Frobenius conjecture: The Frobenius kernel of the Frobenius group was conjectured to be nilpotent. I liked that conjecture for the following reason: If you take the group of proper motions of the Euclidean plane, it is a geometric fact that every proper motion is either a translation or is a rotation. I hope kids are still learning that. It is a curious phenomenon. And the translations form a normal subgroup. So that is something you could actually trace back to antiquity.

No doubt Frobenius knew that. So when he proved his theorem about the existence of the normal complement, that was a link back to very old things to be traced in geometry, I feel. That was one of the appeals. And then the attempt to use Sylow's theorems and a bit of character theory, whatever really, to deal with that problem. That is how I first got really gripped by pure mathematics.

Mathieu discovered the first sporadic simple groups, the Mathieu groups, in the 1860's and 1870's. Why do you think we had to wait one hundred years, before the next sporadic group was found by Janko, after your paper with Feit? Why did it take so long time?

Thompson: Part of the answer would be the flow of history. The attention of the mathematical community was drawn in other directions. I wouldn't say that group theory, certainly not finite group theory, was really at the centre of mathematical development in the 19th century. For one thing, Riemann came along, topology gained and exerted tremendous influence, and as Jacques has mentioned, analysis was very deep and attracted highly gifted mathematicians. It is true, as you mentioned earlier, that Frobenius was there and Burnside; so group theory wasn't completely in the shadows. But there wasn't a lot going on.

Now, of course, there is a tremendous amount going on, both within pure and applied mathematics. There are many things that can attract people, really. So why there was this gap between these groups that Mathieu found and then the rather rapid development in the last half of the 20th century of the simple groups, including the sporadic groups, I have to leave that to the historians. But I don't find it all that mysterious, really. You know, mathematics is a very big subject.

The Feit-Thompson theorem

The renowned Feit–Thompson theorem – finite groups of odd order are solvable - that you proved in the early 1960's: that was originally a conjecture by Burnside, right?

Thompson: Burnside had something about it, yes. And he actually looked at some particular integers and proved that groups of that order were solvable. So he made a start.

When you and Feit started on this project were there any particular results preceding your attack on the Burnside conjecture that made you optimistic about being able to prove it?

Thompson: Sure. A wonderful result of Michio Suzuki, the so-called CA theorem. Absolutely basic! Suzuki came to adulthood just at the end of the Second World War. He was raised in Japan. Fortunately, he came to the University of Illinois. I think it was in 1952 that he published this paper on the CA groups of odd order and proved they were solvable by using exceptional character theory. It is not a very long paper. But it is incredibly ingenious, it seems to me. I still really like that paper. I asked him later how he came about it, and he said he thought about it for two years, working quite hard. He finally got it there. That was the opening wedge for Feit and me, really. The wedge got wider and wider.

Tits: Could you tell me what a CA group is?

Thompson: A CA group is a group in which the centralizer of every non-identity element is abelian. So we can see Abel coming in again: Abelian centralizer, that is what the A means.

The proof that eventually was written down by Feit and you was 255 pages long, and it took one full issue of the Pacific journal to publish.

Thompson: It was long, yes.

It is such a long proof and there were so many threads to connect. Were you nervous that there was a gap in this proof?

Thompson: I guess so, right. It sort of unfolded in what seemed to us a fairly natural way; part group theory, part character theory and this funny little number-theoretic thing at the end. It all seemed to fit together. But we could have made a mistake, really. It has been looked at by a few people since then. I don't lose sleep about it.

It seems that, in particular in finite group theory, there did not exist that many connections to other fields of mathematics like analysis, at least at the time. This required that you had to develop tools more or less from scratch, using ingenious arguments. Is that one of the reasons why the proofs are so long?

Thompson: That might be. It could also be that proofs can become shorter. I don't know whether that will be the case. I certainly can't see that the existing proofs will become tremendously shorter in my lifetime. These are delicate things that need to be explored.

Tits: You see, there are results that are intrinsically difficult. I would say that this is the case of the Feit–Thompson result. I personally don't believe that the proof will be reduced to scratch.

Thompson: I don't know whether it will or not. I don't think mathematics has reached the end of its tether, really.

Tits: It may of course happen that one can go around these very fine proofs, like John's proof, using big machinery like functional analysis. That one suddenly gets a big machine which crushes the result. That is not completely impossible. But the question is whether it is worth the investment.


Jacques Tits

The theory of buildings

Professor Tits, you mentioned already Lie groups as a point of departure. Simple Lie groups had already been classified to a large extent at the end of the 19th century, first by Killing and then by Élie Cartan, giving rise to a series of matrix groups and the five exceptional simple Lie groups. For that purpose, the theory of Lie algebras had to be developed. When you started to work on linear algebraic groups, there were not many tools available. Chevalley had done some pioneering work, but the picture first became clear when you put it in the framework of buildings: this time associating geometric objects to groups. Could you explain us how the idea of buildings, consisting of apartments, chambers, all of these suggestive words, how it was conceived, what it achieved and why it has proven to be so fruitful?

Tits: First of all, I should say that the terminology like buildings, apartments and so on is not mine. I discovered these things, but it was Bourbaki who gave them these names. They wrote about my work and found that my terminology was a shambles. They put it in some order, and this is how the notions like apartments and so on arose.

I studied these objects because I wanted to understand these exceptional Lie groups geometrically. In fact, I came to mathematics through projective geometry: what I knew about was projective geometry. In projective geometry you have points, lines and so on. When I started studying exceptional groups I sort of looked for objects of the same sort. For instance, I discovered — or somebody else discovered, actually — that the group E_6 is the colineation group of the octonion projective plane. And a little bit later, I found some automatic way of proving such results, starting from the group to reconstruct the projective plane. I could use this procedure to give geometric interpretations of the other exceptional groups, e.g., E_7 and E_8. That was really my starting point.

Then I tried to make an abstract construction of these geometries. In this construction I used terms like skeletons, for instance, and what became apartments were called skeletons at the time. In fact, in one of the volumes of Bourbaki, many of the exercises are based on my early work.

An additional question about buildings: This concept has been so fruitful and made connections to many areas of mathematics that maybe you didn't think of at the time, like rigidity theory for instance?

Tits: For me it was really the geometric interpretations of these mysterious groups, the exceptional groups that triggered everything. Other people have then used these buildings for their own work. For instance, some analysts have used them. But in the beginning I didn't know about these applications.

You asked some minutes ago about CA groups. Maybe we can ask you about BN-pairs: what are they and how do they come in when you construct buildings?

Tits: Again, you see, I had an axiomatic approach towards these groups. The BN-pairs were an axiomatic way to prove some general theorems about simple algebraic groups. A BN-pair is a pair of two groups B and N with some simple properties. I noticed that these properties were sufficient to prove, I wouldn't say deep, but far-reaching results; for instance, proving the simplicity property. If you have a group with a BN-pair you have simple subgroups free of charge. The notion of BN-pairs arises naturally in the study of split simple Lie groups. Such groups have a distinguished conjugacy class of subgroups, namely the Borel subgroups. These are the B's of a distinguished class of BN-pairs.

The classification of finite simple groups

We want to ask you, Professor Thompson, about the classification project, the attempt to classify all finite simple groups. Again, the paper by Feit and you in 1962 developed some techniques. Is it fair to say that without that paper the project would not have been doable or even realistic?

Thompson: That I can't say.

Tits: I would say yes.

Thompson: Maybe, but the history has bifurcations so we don't know what could have happened.

The classification theorem for finite simple groups was probably the most monumental collaborative effort done in mathematics, and it was pursued over a long period of time. Many people have been involved, the final proof had 10 000 pages, at least, originally. A group of people, originally led by Gorenstein, are still working on making the proof more accessible.

We had an interview here five years ago with the first Abel Prize recipient Jean-Pierre Serre. At that time, he told us that there had been a gap in the proof, that only was about to be filled in at the time of the interview with him. Before, it would have been premature to say that one actually had the proof. The quasi-thin case was left. How is the situation today? Can we really trust that this theorem finally has been proved?

Thompson: At least that quasi-thin paper has been published now. It is quite a massive work itself, by Michael Aschbacher and Stephen Smith; quite long, well over 1000 pages. Several of the sporadic simple groups come up. They characterize them because they are needed in quasi-thin groups. I forget which ones come up, but the Rudvalis group certainly is among them. It is excruciatingly detailed.

It seems to me that they did an honest piece of work. Whether one can really believe these things is hard to say. It is such a long proof that there might be some basic mistakes. But I sort of see the sweep of it, really. It makes sense to me now. In some way it rounded itself off. I can sort of see why there are probably no more sporadic simple groups; but not really conceptually. There is no conceptual reason that is really satisfactory.

But that's the way the world seems to be put together. So we carry on. I hope people will look at these papers and see what the arguments are and see how they fit together. Gradually this massive piece of work will take its place in the accepted canon of mathematical theorems.

Tits: There are two types of group theorists. Those who are like St. Thomas: they don't believe because they have not seen every detail of the proof. I am not like them, and I believe in the final result although I don't know anything about it. The people who work on or who have worked on the classification theorem may of course have forgotten some little detail somewhere. But I don't believe these details will be very important. And I am pretty sure that the final result is correct.

May we ask about the groups that are associated with your names? You have a group that's called the Thompson group among the sporadic simple groups. How did it pop up? How were you involved in finding it?

Thompson: That is in fact a spin-off from the Monster Group. The so-called Thompson group is essentially the centralizer of an element of order three in the Monster. Conway and Norton and several others were beavering away — this was before Griess constructed the Monster — working on the internal structure where this group came up, along with the Harada–Norton group and the Baby Monster. We were all working trying to get the characters.

The Monster itself was too big. I don't think it can be done by hand. Livingstone got the character table, the ordinary complex irreducible characters of the Monster. But I think he made very heavy use of a computing machine. And I don't think that has been eliminated. That's how the figure 196883 came about, the degree of the smallest faithful complex representation of the Monster Group. It is just too big to be done by hand. But we can do these smaller subgroups.

The Tits group was found by hand, wasn't it? And what is it all about?

Tits: Yes, it was really sort of a triviality. One expects that there would be a group there except that one must take a subgroup of index two so that it becomes simple. And that is what I know about this.

Professor Tits, there is a startling connection between the Monster Group, the biggest of these sporadic groups, and elliptic function theory or elliptic curves via the j-func-

tion. Are there some connections with other exceptional groups, for instance in geometry?

Tits: I am not a specialist regarding these connections between the Monster Group, for instance, and modular functions. I don't really know about these things, I am ashamed to say. I think it is not only the Monster that is related to modular forms, also several other sporadic groups. But the case of the Monster is especially satisfactory because the relations are very simple in that case. Somehow smaller groups give more complicated results. In the case of the Monster, things sort of round up perfectly.

The inverse Galois problem

May we ask you, Professor Thompson, about your work on the inverse Galois problem? Can you explain first of all what the problem is all about? And what is the status right now?

Thompson: The inverse Galois problem probably goes back already to Galois. He associated a group to an equation; particularly to equations in one variable with integer coefficients. He then associated to this equation a well-defined group now called the Galois group, which is a finite group. It captures quite a bit of the nature of the roots, the zeros, of this equation. Once one has the notion of a field, the field generated by the roots of an equation has certain automorphisms and these automorphisms give us Galois groups.

The inverse problem is: Start with a given finite group. Is there always an equation, a polynomial with one indeterminate with integer coefficients, whose Galois group is that particular group? As far as I know it is completely open whether or not this is true. And as a test case if you start with a given finite simple group; does it occur in this way? Is there an equation waiting for it? If there is one equation there would be infinitely many of them. So we wouldn't know how to choose a standard canonical equation associated to this group. Even in the case of simple groups, the inverse problem of Galois Theory is not solved. For the most general finite groups, I leave it to the algebraic geometers or whoever else has good ideas whether this problem is amenable. A lot of us have worked on it and played around with it, but I think we have just been nibbling at the surface.

For example the Monster is a Galois group over the rationals. You can't say that about all sporadic groups. The reason that the Monster is a Galois group over the rationals comes from character theory. It is just given to you.

Tits: This is very surprising: you have this big object, and the experts can tell you that it is a Galois group. In fact, I would like to see an equation.

Is there anything known about an equation?

Thompson: It would have to be of degree of at least 10^{20}. I found it impressive, when looking a little bit at the j-function literature before the days of computers that people like Fricke and others could do these calculations. If you look at the coefficients of the j-functions, they grow very rapidly into the tens and hundreds of millions. They had been computed in Fricke's book. It is really pleasant to see these numbers out there before computers were around. Numbers of size 123 millions. And the numbers had to be done by hand, really. And they got it right.

Tits: It is really fantastic what they have done.

Could there be results in these old papers by Fricke and others that people are not aware of?

Thompson: No, people have gone through them, they have combed through them.

Tits: Specialists do study these papers.

The E_8-story

There is another collaborative effort that has been done recently, the so-called E_8–story: a group of mathematicians has worked out the representations of the E_8. In fact, they calculated the complete character table for E_8. The result has been publicized last year in several American newspapers under the heading "A calculation the size of Manhattan" or something like that.

Thompson: It was a little bit garbled maybe. I did see the article.

Can you explain why we all should be interested in such a result? Be it as a group theorist, or as a general mathematician, or even as man on the street?

Thompson: It is interesting in many ways. It may be that physicists have something to do with the newspapers. Physicists, they are absolutely fearless as a group. Any mathematical thing they can make use of they will gobble right up and put in a context that they can make use of, which is good. In that sense mathematics is a handmaiden for other things. And the physicists have definitely gotten interested in exceptional Lie groups. And E_8 is out there, really. It is one of the great things.

Is there any reason to believe that some of these exceptional groups or sporadic groups tell us something very important - in mathematics or in nature?

Thompson: I am not a physicist. But I know physicists are thinking about such things, really.

Tits: It is perhaps naive to say this: But I feel that mathematical structures that are so beautiful like the Monster must have something to do with nature.

Mathematical work

Are there any particular results that you are most proud of?

Thompson: Well, of course one of the high points of my mathematical life was the long working relationship I had with Walter Feit. We enjoyed being together and enjoyed the work that we did; and, of course, the fusion of ideas. I feel lucky to have had that contact and proud that I was in the game there.

Tits: I had a very fruitful contact for much of my career with François Bruhat and it was very pleasant to work together. It was really working together like you did it, I suppose, with Walter Feit.

Was not Armand Borel also very important for your work?

Tits: Yes, I also had much collaboration with Borel. But that was different in the following sense: when I worked with Borel, I had very often the impression that we both had found the same thing. We just put the results together in order not to duplicate. We wrote our papers practically on results that we had both found separately. Whereas with Bruhat, it was really joint work, complementary work.

Has any of you had the lightning flash experience described by Poincaré; seeing all of a sudden the solution to a problem you had struggled with for a long time?

Tits: I think this happens pretty often in mathematical research; that one suddenly finds that something is working. But I cannot recall a specific instance. I know that it has happened to me and it has happened to John, certainly. So certainly some of the ideas one had works out, but it sort of disappears in a fog.

Thompson: I think my wife will vouch for the fact that when I wake in the morning I am ready to get out there and get moving right away. So my own naïve thinking is that while I am asleep there are still things going on. And you wake up and say: "Let's get out there and do it." And that is a wonderful feeling.

You have both worked as professors of mathematics in several countries. Could you comment on the different working environments at these places and people you worked with and had the best cooperation with?

Tits: I think the country which has the best way of working with young people is Russia. Of course, the French have a long tradition and they have very good, very young people. But I think Russian mathematics is in a sense more lively than French mathematics. French mathematics is too immediately precise. I would say that these are the two countries where the future of mathematics is the clearest. But of course Germany has had such a history of mathematics that they will continue. And nowadays, the United States have in a sense become the centre of mathematics, because they have so much money. That they can...

…buy the best researchers?

Tits: That's too negative a way of putting it. Certainly many young people go the United States because they cannot earn enough money in their own country.

And of course the catastrophe that happened in Europe in the 1930's with Nazism. A lot of people went to the United States.

What about you, Professor Thompson? You were in England for a long time. How was that experience compared to work at an American university?

Thompson: Well, I am more or less used to holding my own role. People didn't harass me very much any place. I have very nice memories of all the places I have visited, mainly in the United States. But I have visited several other countries, too, for shorter periods, including Russia, Germany and France. Mathematically, I feel pretty much comfortable everywhere I am. I just carry on. I have not really been involved in higher educational decision making. So in that sense I am not really qualified to judge what is going on at an international basis.

Thoughts on the development of mathematics

You have lived in a period with a rapid development of mathematics, in particular in your own areas, including your own contributions. Some time ago, Lennart Carleson, who received the Abel Prize two years ago, said in an interview that the 20th century had possibly been the Golden Age of Mathematics, and that it would be difficult to imagine a development as rapid as we have witnessed it.
What do you think: Have we already had the Golden Age of Mathematics or will development continue even faster?

Tits: I think it will continue at its natural speed, which is fast; faster than it used to be.

Thompson: I remember reading a quote attributed to Laplace. He said that mathematics might become so deep, that we have to dig down so deep, that we will not be able to get down there in the future. That's a rather scary image, really. It is true that prerequisites are substantial but people are ingenious. Pedagogical techniques might change. Foundations of what people learn might alter. But mathematics is a dynamic thing. I hope it doesn't stop.

Tits: I am confident that it continues to grow.

Traditionally, mathematics has been mainly linked to physics. Lots of motivations come from there, and many of the applications are towards physics. In recent years, biology, for example with the Human Genome Project, economics with its financial mathematics, computer science and computing have been around, as well. How do you judge these new relations? Will they become as important as physics for mathematicians in the future?

Tits: I would say that mathematics coming from physics is of high quality. Some of the best results we have in mathematics have been discovered by physicists. I am less sure about sociology and human science. I think biology is a very important subject but I don't know whether it has suggested very deep problems in mathematics. But perhaps I am wrong. For instance, I know of Gromov, who is a first class mathematician, and who is interested in biology now. I think that this is a case where really mathematics, highbrow mathematics, goes along with biology. What I said before about sociology, for instance, is not true for biology. Some biologists are also very good mathematicians.

Thompson: I accept that there are very clever people across the intellectual world. If they need mathematics they come up with mathematics. Either they tell mathe-

maticians about it or they cook it up themselves.

Thoughts on the teaching of mathematics

How should mathematics be taught to young people? How would you encourage young people to get interested in mathematics?

Thompson: I always give a plug for Gamow's book One Two Three … Infinity and

Courant and Robbins' What is Mathematics? and some of the expository work that you can get from the libraries. It is a wonderful thing to stimulate curiosity. If we had recipes, they would be out there by now. Some children are excited, and others are just not responsive, really. You have the same phenomenon in music. Some children are very responsive to music, others just don't respond. We don't know why.

Tits: I don't know what to say. I have had little contact with very young people. I have had very good students, but always advanced students. I am sure it must be fascinating to see how young people think about these things. But I have not had the experience.

Jean-Pierre Serre once said in an interview that one should not encourage young people to do mathematics. Instead, one should discourage them. But the ones that, after this discouragement, still are eager to do mathematics, you should really take care of them.

Thompson: That's a bit punitive. But I can see the point. You try to hold them back and if they strain at the leash then eventually you let them go. There is something to it. But I don't think Serre would actually lock up his library and not let the kids look at it.

Maybe he wants to stress that research mathematics is not for everyone.

Thompson: Could be, yeah.

Tits: But I would say that, though mathematics is for everyone, not everyone can do it with success. Certainly it is not good to encourage young people who have no gift to try to do something, because that will result in sort of a disaster.

Personal interests

In our final question we would like to ask you both about your private interests besides mathematics. What are you doing in you spare time? What else are you interested in?

Tits: I am especially interested in music and, actually, also history. My wife is a historian; therefore I am always very interested in history.

What type of music? Which composers?

Tits: Oh, rather ancient composers.

And in history: Is that old or modern history?

Tits: Certainly not contemporary history, but modern and medieval history. All related to my wife's speciality.

Thompson: I would mention some of the same interests. I like music. I still play the piano a bit. I like to read. I like biographies and history; general reading, both contemporary and older authors. My wife is a scholar. I am interested in her scholarly achievements. Nineteenth century Russian literature; this was a time of tremendous achievements. Very interesting things! I also follow the growth of my grandchildren.

Tits: I should also say that I am very interested in languages; Russian, for instance.

Do you speak Russian?

Tits: I don't speak Russian. But I have been able to read some Tolstoy in Russian. I have forgotten a little. I have read quite a lot. I have learned some Chinese. In the course of years I used to spend one hour every Sunday morning studying Chinese. But I started a little bit too old, so I forgot what I learned.

Are there any particular authors that you like?

Tits: I would say all good authors.

Thompson: I guess we are both readers. Endless.

Let us finally thank you very much for this pleasant interview; on behalf of the Norwegian, the Danish and the European Mathematical Societies. Thank you very much.

Thompson: Thank you.

Tits: Thank you for the interview. You gave us many interesting topics to talk about!

MatematikerNyt

ved Mikael Rørdam

Andreas Klose is as of January 1, 2007, employed as a lektor in Operations Research at the Mathematics Department at Aarhus University.

Andreas studied Economics at the University of Wuppertal in Germany, where he graduated in 1989 with a thesis on bound-and-scan algorithms in integer programming. He then moved to the Business School in St. Gallen (Switzerland), where he completed a Ph.D. study in 1993 with a doctoral thesis on the "discrete p-median problem". From 1993-2000 he worked as a lecturer and researcher at the Institute for Operations Research, University of St. Gallen, and completed in 2000 his habilitation thesis on "location planning in distribution systems". He then joined the Institute for Operations Research and Mathematical Methods in Economics at the University of Zurich until he finally came to Aarhus.

Andreas is doing research and teaching in Operations Research and Logistics, in particular on combinatorial optimisation and integer programming with applications to Logistics.

His private hobbies are cycling and swimming.

Helle Sørensen er pr. 1. oktober 2008 ansat som lektor ved Institut for Matematiske Fag, Københavns Universitet.

Helle er Cand. Scient. (1997) og phd (2000) i statistik fra Københavns Universitet, og kommer fra en lektorstilling på KU-LIFE. Helle arbejder med statistik for stokastiske processer, især processer baseret på stokastiske differentialligninger, og har desuden arbejdet med anvendelser indenfor især toksikologi og veterinærvidenskab.

Bergfinnur Durhuus er med tiltrædelse d. 1. september 2008 ansat som professor i Matematisk Fysik ved Institut for Matematiske Fag, Københavns Universitet.

Bergfinnur er uddannet Cand. Scient. i matematik og fysik fra Københavns Universitet (1976) og gennemførte sit Ph.D. studium ved Institut des Hautes Etudes i Paris og ved NORDITA i København. Han har været ansat som lektor ved Københavns Universitet siden 1988 og i perioder som gæsteforsker ved. bl.a. Yukawa Institute for Theoretical Physics, Kyoto, og Institut Elie Cartan, Nancy.

Bergfinnurs forskning omfatter flere områder af Matematisk Fysik, herunder Kvantefeltteori, Statistisk Mekanik, Strengteori, Kvantegeometri, Topologisk Kvantefeltteori og Ikke-kommutativ Feltteori. Specielt har han i samarbejde med andre indført de såkaldte »dynamisk triangulerede« modeller for strenge og to-dimensional kvantegravitation, som har udviklet sig til et vigtigt middel til forståelse af Konform Feltteori og dens samspil med Strengteori og med teorien for stokastiske matricer.

Mogens Steffensen er ansat om professor (mso) ved Institut for Matematiske Fag, Københavns Universitet med tiltrædelse 1. oktober 2008. Mogens er cand. act. (kandidat i forsikringsmatematik) (1997) og Ph.D. i forsikringsmatematik (2001) fra Københavns Universitet. Her har han tillige været ansat som adjunkt (2001-2004) og lektor (2004-2008). Undervejs har han haft længere forskningsophold ved Karlsruhe Universitet (1999), Stony Brook New York (2000), Kaiserslautern Universitet (2003) og London School of Economics (2005).

Hans forskning handler om integrationen af finansiel matematik og forsikringsmatematik, specielt med anvendelse af stokastisk kontrol inden for optimale beslutningsproblemer i livs- og pensionforsikring.

Fritiden bruger han på familie, venner, bøger og film. Herudover spiller han gerne 'alt muligt' i sit pop-reggaeband – keyboard, saxofon, trompet, og kor.

Søren Eilers er med tiltrædelse 1. marts 2008 ansat som professor med særlige opgaver ved Institut for Matematiske Fag, Københavns Universitet hvor han også virker som viceinstitutleder for undervisning med ansvar for kvalitet og udvikling af instituttets undervisningsudbud, og hvor han har været ansat siden 1997. Søren har indtil foråret været aktiv i foreningens bestyrelse som først kasserer og siden formand.

Sørens forskningsområde er operatoralgebraen med særlig fokus på klassifikation med K-teoretiske invarianter. Dette emne har ført ham til en sideinteresse for symbolske dynamiske systemer hvis klassifikationsteori er tæt knyttet til operatoralgebraens, og til mere funktionanalytiske aspekter knyttet til de løfteproblemer der har central teknisk betydning her.

Sørens børn Lea og Andreas er i en alder der ikke giver ham så gode muligheder for at dyrke fritidsinteresser, men han sætter pris på en god bog og på at spille Scrabble... især når han vinder!

Aftermath

ved Mogens Esrom Larsen

LØSNINGER

To af opgaverne i sidste nr. er hentet fra Frederick Mosteller, Fifty Challenging Problems in Probability with Solutions, Dover, New York 1987.

Ebbe Thue Poulsen og problemklubben "Con Amore" har løst de to opgaver herfra.

En tryllekunst

Tryllekunstneren og hans assistent præsenterer publikum for 8 mønter på en række. Tryllekunstneren instruerer publikum om opgaven og forlader lokalet. Publikum vælger nu for hver mønt, om den skal være krone eller plat. Derefter oplyser publikum assistenten om deres foretrukne mønt, fx nr 5 fra venstre. Nu vender assistenten én af mønterne om efter sit valg.

Tryllekunstneren kommer ind fra kulissen og udpeger den foretrukne mønt.

Vi stiller mønterne på række og giver dem numre fra venstre mod højre, 0, 1, ..., 7. Disse tal organiserer vi som gruppen $\mathbb{Z}_2^3$, fx ved at skrive numrene binært fra 000 til 111 og definere gruppeoperationen som addition uden mente – eller, om man vil, med regnereglen $1+1=0$. Så fx er $3+6=011+110=101=5$. En række af mønter gives nu værdien, der er summen af numrene på de viste kroner. Mønsteret PPKKPKPP får således summen 2+3+5=010+011+101=100=4. Hvis nu publikum vælger at pege på mønt nr 3, så skal vi vende en mønt, så summen bliver 3 i stedet for 4. Vi skal altså løse ligningen $4+x=3$. Men da alle elementer i gruppen er deres egen inverse, er $x=3+4=011+100=111=7$. Vi skal derfor vende den sidste mønt, så vi får PPKKPKPK med summen 2+3+5+7=010+011+101+111=011=3.

Dette trick virker for enhver potens af 2, men med andre antal mønter kan kunsten ikke udføres. Prøv selv at lave tryllekunsten med 3 mønter!

En sum

I Amer. Math. Monthly April 2008 stilles som problem 11356 en opgave af Michael Poghosyan, Yerevan State University, Yerevan, Armenien.

Vis identiteten

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}$$

Bevis:

Vi definerer den nedstigende faktoriel med angivet skridtlængde således

$$[x,d]_n := \begin{cases} \prod_{j=0}^{n-1}(x-jd) & n \in \mathbb{N} \\ 1 & n = 0 \\ \prod_{j=1}^{-n} \dfrac{1}{x+jd} & -n \in \mathbb{N},\ -x \notin \{d, 2d, \cdots, -nd\} \end{cases}$$

Vi omskriver de fleste binomialkoefficienter i summen til faktorieller

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n!(2k)!(2n-2k)!}{k!(n-k)!(2n)!(2k+1)}$$

Faktoriellerne med et 2–tal deles i faktorieller af hvert andet led og skridtlængde 2

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n!\,[2k,2]_k\,[2k-1,2]_k\,[2n-2k,2]_{n-k}\,[2n-2k-1,2]_{n-k}}{k!(n-k)!\,[2n,2]_n\,[2n-1,2]_n\,(2k+1)}$$

Nu halveres alle faktorerne, så skridtlængden bliver 1, med korrektioner af diverse potenser af 2

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n!k!2^k\left[k-\frac{1}{2},1\right]_k 2^k(n-k)!2^{n-k}\left[n-k-\frac{1}{2},1\right]_{n-k} 2^{n-k}}{k!(n-k)!n!2^n\left[n-\frac{1}{2},1\right]_n 2^n(2k+1)}$$

Efter at have forkortet, hvad som kan, fås

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left[k-\frac{1}{2},1\right]_k\left[n-k-\frac{1}{2},1\right]_{n-k}}{\left[n-\frac{1}{2},1\right]_n(2k+1)} = \frac{1}{\left[n-\frac{1}{2},1\right]_n}\sum_{k=0}^{n} \binom{n}{k} \frac{\left[k-\frac{1}{2},1\right]_k\left[n-k-\frac{1}{2},1\right]_{n-k}}{2k+1}$$

For at komme nævneren til livs indføres faktoriellen

$$\left[n+\tfrac{1}{2},1\right]_n\cdot\frac{1}{2} = \left[n+\tfrac{1}{2},1\right]_{n+1} = \left[n+\tfrac{1}{2},1\right]_{n-k}\left(k+\tfrac{1}{2}\right)\left[k-\tfrac{1}{2},1\right]_k$$

hvorved fås

$$\frac{1}{2k+1} = \frac{\left[n+\frac{1}{2},1\right]_{n-k}\left[k-\frac{1}{2},1\right]_k}{\left[n+\frac{1}{2},1\right]_n}$$

Så kan vi skrive

$$\frac{1}{\left[n-\frac{1}{2},1\right]_n\left[n+\frac{1}{2},1\right]_n}\sum_{k=0}^{n} \binom{n}{k}\left[k-\tfrac{1}{2},1\right]_k\left[n-k-\tfrac{1}{2},1\right]_{n-k}\left[n+\tfrac{1}{2},1\right]_{n-k}\left[k-\tfrac{1}{2},1\right]_k$$

Nu skifter vi fortegn i alle faktorerne i de faktorieller, der indeholder et k i starten

$$\frac{1}{\left[n-\frac{1}{2},1\right]_n\left[n+\frac{1}{2},1\right]_n}\cdot$$
$$\sum_{k=0}^{n} \binom{n}{k}\left[-\tfrac{1}{2},1\right]_k(-1)^k\left[-\tfrac{1}{2},1\right]_{n-k}(-1)^{n-k}\left[n+\tfrac{1}{2},1\right]_{n-k}\left[-\tfrac{1}{2},1\right]_k(-1)^k$$

hvilket skrives pænere som

$$\frac{(-1)^n}{\left[n-\frac{1}{2},1\right]_n\left[n+\frac{1}{2},1\right]_n}\sum_{k=0}^{n} \binom{n}{k}\left[-\tfrac{1}{2},1\right]_k^2\left[-\tfrac{1}{2},1\right]_{n-k}\left[n+\tfrac{1}{2},1\right]_{n-k}(-1)^k$$

Dette udtryk genkendes som Pfaff-Saalschütz' formel, (9.1), i min nylige lærebog, *Summa Summarum,* A K Peters 2007:

Theorem 9.1. *If the – complex – numbers satisfy $a_1 + a_2 + b_1 + b_2 = n - 1$ we have the Pfaff-Saalschütz formula (J. F. Pfaff 1797, L. Saalschutz 1890)*

$$\sum_{k=0}^{n} \binom{n}{k} [a_1, 1]_k [a_2, 1]_k [b_1, 1]_{n-k} [b_2, 1]_{n-k} (-1)^k = [b_1 + a_1, 1]_n [b_1 + a_2, 1]_n (-1)^n$$

Så vi kan reducere til

$$\frac{1}{\left[n-\frac{1}{2},1\right]_n \left[n+\frac{1}{2},1\right]_n} [-1,1]_n^2 = \frac{n!^2}{\left[n-\frac{1}{2},1\right]_n \left[n+\frac{1}{2},1\right]_n} =$$
$$= \frac{2^{2n} n!^2}{[2n-1,2]_n [2n+1,2]_n} = \frac{2^{4n} n!^4}{[2n-1,2]_n [2n,2]_n^2 [2n+1,2]_n} = \frac{2^{4n} (n!)^4}{(2n)!(2n+1)!}$$

Sokker, der passer til hinanden

Når sandsynligheden for at få to røde sokker er $\frac{1}{2}$, når man trækker to tilfældigt ud af en sæk med røde og sorte sokker, hvor mange er der så af hver farve i sækken?

Det samlede antal sokker betegnes med m, og antallet af røde sokker med n. Så kan man udtage et par sokker på $m(m-1)/2$ måder, og et par røde sokker på $n(n-1)/2$ måder.

Sandsynligheden for, at et tilfældigt udtaget par sokker er røde, er altså $n(n-1)/m(m-1)$, og denne sandsynlighed er $\frac{1}{2}$, hvis og kun hvis

$$m(m-1) = 2n(n-1)\,. \tag{1}$$

Et talpar (m, n), som er løsning til (1), giver en løsning til sokkeproblemet, hvis m og n er hele tal ≥ 2. Ligningen (1) kan omformes til

$$(2m-1)^2 = 2(2n-1)^2 - 1\,,$$

og det ses let, at hvis (x, y) er en heltallig løsning til

$$x^2 - 2y^2 = -1\,, \tag{2}$$

så er x og y begge ulige, således at $m = (x+1)/2$ og $n = (y+1)/2$ er hele. For at finde samtlige løsninger til sokkeproblemet, skal vi altså finde samtlige heltallige løsninger til (2) med $x \geq 3$ og $y \geq 3$.

For reelle tal z i ringen $\mathbb{Z}\left[\sqrt{2}\right]$ af tal af formen $z = x + y\sqrt{2}$ med $x, y \in \mathbb{Z}$ indfører vi betegnelsen $z = x - y\sqrt{2}$.

Vi bemærker, at der for $z, w \in \mathbb{Z}\left[\sqrt{2}\right]$ gælder $zw = z\,w$, samt at (x, y) er løsning til (2), hvis og kun hvis $z = x + y\sqrt{2}$ er løsning til

$$zz = -1\,. \tag{3}$$

Idet vi sætter $e = 1 + 1\sqrt{2}$, ser vi, at $ee = -1$, hvoraf følger, at z er løsning til (3), hvis og kun hvis $e^2 z$ er det.

Hermed har vi uendeligt mange løsninger til (3), nemlig

$$z_k = e^{2k+1}, \quad k = 0, 1, 2, \dots . \tag{4}$$

Her giver $z_0 = e = 1 + 1\sqrt{2}$ ikke nogen løsning til sokkeproblemet, men alle z_k med $k \geq 1$ gør, og specielt har vi $z_1 = 7 + 5\sqrt{2}$, der giver $(m_1, n_1) = (4, 3)$.

Ved brug af binomialformlen i (4), får vi

$$x_k = \sum_{j=0}^{k} \binom{2k+1}{2j+1} 2^{k-j} , \quad y_k = \sum_{j=0}^{k} \binom{2k+1}{2j} 2^{k-j} ,$$

der for $k \geq 1$ giver løsningen

$$m_k = \sum_{j=0}^{k-1} \binom{2k+1}{2j+1} 2^{k-j-1} + 1 , \quad n_k = \sum_{j=0}^{k-1} \binom{2k+1}{2j} 2^{k-j-1} + k + 1 ,$$

til sokkeproblemet.

Da $z_k = e^2 z_{k-1} = (3 + 2\sqrt{2}) z_{k-1}$, kan følgen $\{(x_k, y_k)\}_{k=1}^{\infty}$ også bestemmes rekursivt ved

$$\begin{aligned} &x_1 = 7, && y_1 = 5 , \\ &x_k = 3x_{k-1} + 4y_{k-1} , && y_k = 2x_{k-1} + 3y_{k-1} && \text{for } k \geq 2 . \end{aligned}$$

Ved indsættelse af $x_k = 2m_k - 1$, $y_k = 2n_k - 1$ heri fås rekursionligningerne

$$\begin{aligned} &m_1 = 4 , && n_1 = 3 , \\ &m_k = 3m_{k-1} + 4n_{k-1} - 3 , && n_k = 2m_{k-1} + 3n_{k-1} - 2 && \text{for } k \geq 2 \end{aligned}$$

til bestemmelse af løsninger til sokkeproblemet.

Jeg vil slutte med at bevise, at dette er samtlige løsninger. Dertil vil jeg vise, at talsættene $\{(x_k, y_k)\}_{k=0}^{\infty}$ er samtlige positive heltalsløsninger til (2). Lad nemlig (x, y) være en vilkårlig positiv heltalsløsning til (2), sæt $z = x + y\sqrt{2}$, og bemærk, at $z \geq e$.

Lad $k \geq 0$ være bestemt således, at

$$e \leq z e^{-2k} < e^3 , \tag{5}$$

og sæt $z' = z e^{-2k}$.

Da z' kan skrives $z' = z e^{2k}$, er $z' \in \mathbb{Z}\left[\sqrt{2}\right]$, lad os sige $z' = x' + y'\sqrt{2}$.

Da $z'z' = zz = -1$ og $z' > 1$, er $|z'| = |x' - y'\sqrt{2}| < 1$, hvilket kun kan være opfyldt, hvis x' og y' har samme fortegn, dvs hvis x' og y' begge er positive.

Da ${x'}^2 = 2{y'}^2 - 1$, er $x' = y' = 1$ en mulighed, medens $y' = 2, 3$ eller 4 ikke kan bruges. $y' \geq 5$ giver $x' \geq 7$, og altså $z' = x' + y'\sqrt{2} \geq e^3$ i modstrid med (5). Der må altså gælde $z' = e$, og dermed $z = z' e^{2k} = z_k$.

Travle duellanter

Duellerne i Travløse er sjældent fatale. Hver kombattant møder op på et tilfældigt tidspunkt mellem 5 og 6 om morgenen på den aftalte dag, venter 5 min på sin modstander, og går igen, hvis denne ikke er mødt op. Ellers slås de to.

Hvad er sandsynligheden for, at det kommer til kamp?

Lad (x, y) betegne ankomsttidspunkterne for de to duellanter, regnet i timer fra klokken 5. Så er sandsynlighedsfordelingen for (x, y) Lebesguemålet på enhedskvadratet $[0,1] \times [0,1]$.

Det kommer til kamp, hvis og kun hvis $|y - x| \leq 1/12$, dvs hvis og kun hvis (x, y) *ikke* ligger i en af trekanterne

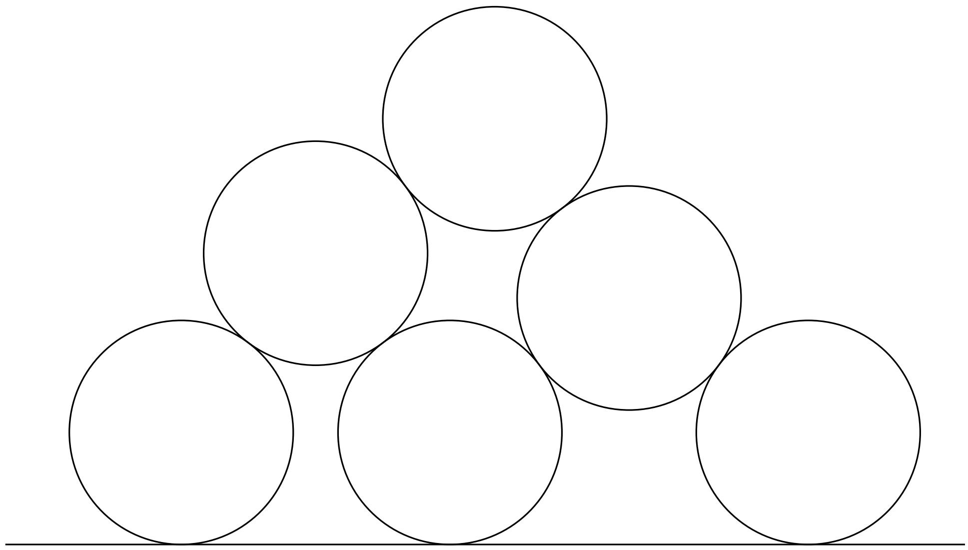
$$T_1: \quad \frac{1}{12} < x \leq 1 \quad \text{og} \quad 0 \leq y < x - \frac{1}{12},$$
$$T_2: \quad 0 \leq x < 1 - \frac{1}{12} \quad \text{og} \quad x + \frac{1}{12} < y \leq 1.$$

Sandsynligheden for, at de to duellanter mødes, er altså

$$1 - 2 \times \frac{1}{2}\left(\frac{11}{12}\right)^2 = \frac{23}{144}.$$

NYE OPGAVER

De lystige kroner



De lystige kroner

Der ligger seks kroner i énkronestykker på et bord. De tre ligger på en ret linie. De ligger ikke helt tæt sammen men så tæt at der ikke kan presses en krone til mellem to af dem.

To kroner støder til to af de tre til samme side og udenpå dem støder den sidste krone til begge de to. Nu kunne det se ud til at den sidste krone ligger lige langt fra de to yderste. Det gør den selvfølgelig hvis den tredie krone ligger lige midt mellem de to andre. Men gør den det altid?

Den mystiske pyramide

Ægyptens ældste pyramide, trinpyramiden ved Sakkara, ligner ikke Cheops' og de andre. Som navnet antyder, har den snarere form som en kæmpetrappe, mere som Mayaernes pyramider.

Hvis vi begynder fra over, er der én sten i det øverste lag, fire i det næste, ni i det tredie, osv.

Når vi nu får at vide, at antallet af sten, der ialt er medgået til byggeriet, er et kvadrattal, og at der er medgået mere end én sten, hvor mange trin har så pyramiden?

(Det er vanskeligt at bevise, at der kun er én løsning.)

Klassikeren

Et sted ude i tundraen havde man fået rejst tre kraftværker, et elektricitetsværk, et gasværk og et vandværk. Nu var det ellers et øde område, der var ialt kun tre huse, der skulle forsynes med strøm, gas og vand.

Men det var ikke helt problemfrit. Man skulle jo trække ledningerne oven på jorden, som var for stivfrossen til at grave i, men ledningerne tålte ikke at krydse hinanden. Og selv om der kun var de tre værker og de tre huse, havde ingeniørerne endnu ikke fundet en rørføring, der løste problemet.

Hvorfor ikke?

Men oppe på den ringformede asteroide, Torus II, var det lykkedes de lokale ingeniører at løse rørføringsproblemet.

Hvordan?

Æventyret

Der var engang en prins, der skulle vælge sig en prinsesse. Han havde valget mellem tre søstre, som alle var unge og smukke. Deres far var en viis gammel konge, og han ville sikre sig, at hans kommende svigersøn havde omløb i hovedet. Så han sagde til prinsen:

"Før du får min velsignelse til at ægte en af mine døtre, vil jeg sætte dit mod og din intelligens på en prøve.

Du får lov til at stille én af prinsesserne ét spørgsmål, som kan besvares med "ja" eller "nej". Den ene vil svare sandfærdigt, den anden vil svare falsk, og den tredje, som er min yndlingsdatter, kan svare sandfærdigt eller falsk, som hun vil. Hun har alligevel aldrig rettet sig efter mig.

Ud fra svaret på dit spørgsmål skal du vælge din brud. Men jeg advarer dig: Hvis du vælger min yndlingsdatter, skal du have dit hoved hugget af!"

Prinsen havde ingen anelse om, hvem der var kongens yndligsdatter, lige så lidt som han anede, hvem der ville tale sandt, og hvem falsk. Han måtte altså formulere sit spørgsmål sådan, at ligegyldigt hvem han spurgte, og ligegyldigt, hvad hun svarede, skulle han ud fra svaret kunne vælge en af de to andre til sin brud.

Naturligvis stillede prinsen et så snedigt spørgsmål, at han med sikkerhed undgik yndlingsprinsessen. Og kongen blev så imponeret, at han alligevel gav prinsen yndlingsdatteren, og de to levede lykkeligt til deres dages ende.

Hvordan mon prinsen formulerede sit spørgsmål?

En rørende historie

Et vandrør er 6,4 cm i diameter, og midt på røret er der et T–rør, så siderøret er 2,7 cm i diameter. Siderøret sidder altså nøjagtig vinkelret på hovedrøret.

Nu løber vandet i en strøm gennem hovedrøret, og en del af vandet løber ud ad siderøret. Man har nu tilsat nogle mikadopinde til vandet, og det er meningen, at de ikke må løbe ud ad siderøret.

Man har derfor spurgt kommuneingeniøren, hvor lange mikadopindene skal være, for at de ikke på nogen måde kan dreje om ad siderøret. Hvis de prøver, skal de sætte sig fast.

Hvad er den kritiske grænseværdi for mikadopindene?

Et biproblem

Betragt en regulær sekskant, der er gennemskåret i et regelmæssigt trekantet mønster. Man tænker sig, at hver side er delt i n lige store stykker, og derefter er alle de linier, der erparallelle med siderne, tegnet.

Problemet er at tælle alle forekommende regulære sekskanter på figuren.

Vejerboden

I den klassiske opgave er der givet 12 kugler, hvoraf de 11 er ens. Man skal så afsløre den aparte i 3 vejninger. Samtidig skal det afgøres, om den er lettere eller tungere end de andre.

Til hjælp har man en almindelig skålvægt med to skåle.

Men til variation af temaet har vi denne gang 14 kugler, hvoraf de 13 vejer nøjagtig 10 g. Desuden har vi et 10 gramslod.

Vi skal igen afsløre den aparte kugle i højst 3 vejninger, men det er ikke krævet, at vi finder ud af, om den er lettere eller tungere end de andre.

Hvordan skal man bære sig ad med det?

Gitterpunkterne

Forleden dag sad jeg og slog krusseduller på et almindeligt ark ternet papir. Så kom jeg for skade at lege med gitterpunkterne. Jeg valgte 5 af dem tilfældigt ud.

Så tegnede jeg alle 10 forbindelseslinier mellem dem. Og hver gang var der et af liniestykkerne, der passerede hen over et gitterpunkt.

Hvorfor det?

Pythagoras

En Pythagoræisk trekant med heltallige sider, x, y og z, der opfylder

$$x^2 + y^2 = z^2$$

må have mindst én side som et lige tal. Og ingen Pythagoræisk trekant har en side af længde 2. Men man kan tænke sig en Pythagoræisk trekant, hvis sider er to primtal og et tal, der er det dobbelte af et primtal.

Opgaven går ud på at bestemme *samtlige* Pythagoræiske trekanter af den slags.

De logiske frimærkesamlere

Tre personer – A, B og C – var alle fuldstændig logiske. De kunne alle tre øjeblikkelig drage alle de logiske konsekvenser af alle præmisser. Desuden vidste hver af dem, at de to andre var lige så logiske som han selv. Man viste dem syv frimærker; to røde, to gule og tre grønne. Derpå fik de bind for øjnene, og et frimærke blev klistret i panden af dem hver især, mens de resterende frimærker blev lagt ned i en skuffe. Da øjenbindene var fjernet, spurgte man A: ”Kan du nævne én farve, som dit frimærke i hvert fald ikke har?” ”Nej,” svarede A. Så fik B det samme spørgsmål, og han svarede også ”nej”.

Er det muligt ud fra disse oplysninger at regne sig frem til, hvilken farve A's frimærke havde? Eller B's? Eller C's?

Joakim von And i Sahara

Joakim von And er som bekendt verdens rigeste og nærigste and. Da han derfor engang skulle køre over Sahara i jeep, måtte han jo spekulere på, hvor billigt det kunne lade sig gøre.

Nu var hans jeeps kun i stand til at køre en trediedel af vejen på en fuld tank, men til gengæld kunne alle hans jeeps køre fuldautomatisk uden chauffør, og han havde masser af dem. Og han kunne let tømme og fylde tankene midt i ørkenen uden at spilde. Men med fuld tank menes så meget benzin, som en jeep på nogen måde kan medbringe.

Problemet er, hvordan slipper Joakim von And billigst muligt over ørkenen, når hele hans flåde af jeeps står på den ene side. Hvor mange jeeps skal han bruge, og hvordan skal han bære sig ad? (Han kan bare efterlade sine jeeps i ørkenen, de skal ikke returneres.)

Ved uanbringelighed returneres bladet til afsender:

Matilde
Institut for Matematiske Fag
Aarhus Universitet
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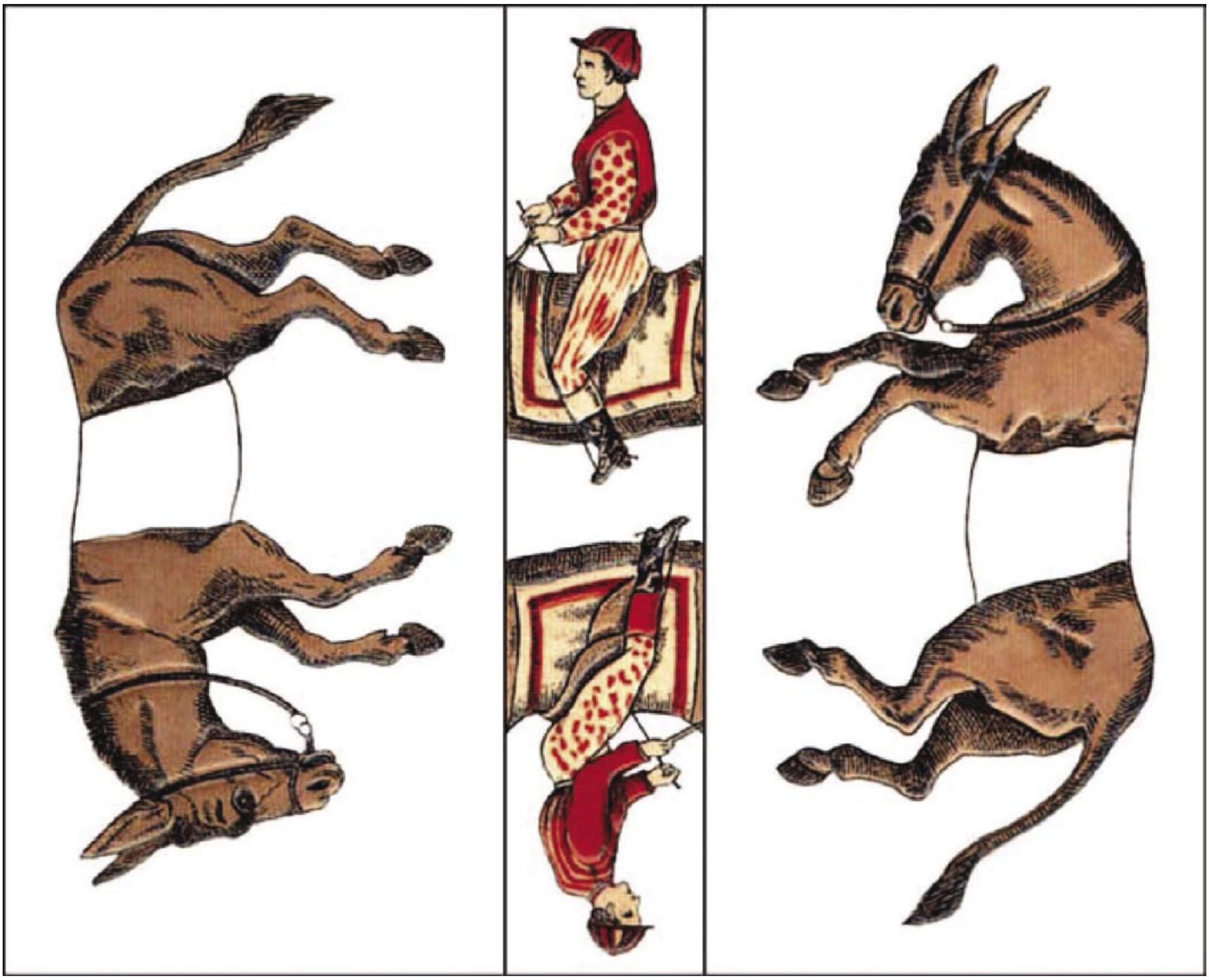
8000 Århus C

POST B
PP DANMARK



Sam Loyd's donkey puzzle. Man skal skære tre rektangler ud og samle stykkerne, således at de 2 jockey'er rider på de to æsler. Løsningen kan findes her: http://www.defectiveyeti.com/mules/mules-solution.jpg